STRUCTURAL MECHANICS OF DEFORMATION AND FRACTURE

FINAL REPORT

ON

CONTRACT NASW 1190

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THE MECHANICS OF DEFORMATION AND FRACTURE

The work described in this final report on NASW-1190 is a continuation of a study initiated under NASW-708 relating to the mechanical behavior of materials which have temperature dependent properties. It was shown in the earlier work that the strengths of such materials are limited by the regenerative thermal feedback that develops because the deformation process is at once exothermic and temperature sensitive.

The analysis of simple model experiments also led to the expectation of effects of sample size, duration of load, rate of strain and rate of loading similar to those observed in real experiments. Furthermore, the introduction of the temperature dependence into the continuum theory provided a bridge to the atomic scale theory of materials through the concept of the energy of activation. Subsequent to the final report on NASW-708, the work was summarized in three articles that have since been published (1), (2), (3).

In the current work, improved methods of analysis have been developed and more realistic model experiments have been studied. In particular, the dynamics of simple deformations of homogeneous and heterogeneous model materials have been considered. Stick-slip effects characteristic of real ductile materials have been demonstrated.

Plastic flow has been simulated and a model designed to relate the gross features of the deformation to crystal defects in the material were studied.

The new analytical schemes that have been introduced involve the use of improved, versatile electric analogs of the mechanical models. In these, the temperature dependent viscosity in the model materials is simulated by thermistor elements for which the temperature dependence of the resistance can be described in terms of an energy of activation.

In the new analogs, the inductors and capacitors that ordinarily simulate the inertial and elastic effects in the material are further simulated by precision electronic integrators. This eliminates the problems of obtaining matched components with a wide range of values. Furthermore, the simulated inductors are linear and have no resistance; qualities that are unavailable in real inductors. In addition, the effective values of these components can readily be made responsive to the local values of the stress, strain or strain rate.

As a by-product of the earlier work, some insights relating to the stability of laminar flow were provided. The current work suggests an approach to the study of earthquakes and volcanism as well as to certain meteorological problems.

The work on NASW-1190 has been summarized in five technical articles that have been submitted for publication. One of these has already appeared (5). Two others have been accepted and seen by the author in galley proof (4, 7). No final decision has been received on the remaining two articles (6), (8). It is a pleasure to report that parts of these studies are being applied to the Ph.D. Thesis of Mr. George Mueller, whose candidacy at the University of Delaware is under the supervision of Professor W. F. Ames. In addition, Dr. Gruntfest was invited to present a review of this work before the British Society of Rheology.

The body of this report consists of copies of the five cited articles. These are separated from one another by colored spacer pages. Their titles are listed below (4 - 8).

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2. "Thermal Effects in Model Viscoelastic Solids," I. J. Gruntfest and S. J. Becker, Transactions Society of Rheology 9:2, 103-119 (1965).
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5. "Thermistor Analogs for Model Viscous and Viscoelastic Systems," I. J. Gruntfest, Transactions Society of Rheology 9:2, 213,225 (1965).

6. "Thermistor Analog Study of Dynamic Shear in Model Viscoelastic Materials," by I. J. Gruntfest and R. C. Good, Jr., Submitted to International Journal of Non-Linear Mechanics.

7. "Thermistor Analog Study of Dynamic Shear in an Ideal Viscous Material," I. J. Gruntfest and G. E. Mueller, accepted for publication by Transactions Society of Rheology (1966).

8. "Analog Study of the Dynamics of a Non-Linear Maxwell Model Material," I. J. Gruntfest and G. E. Mueller, Submitted to Transactions Society of Rheology.

Heat Transfer Considerations in Studies of Mechanical Behavior

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Abstract

As part of a study of the mechanical behavior of materials having temperature dependent properties, the responses of a simple model composite material are examined. The size and distribution of the components determine the thermal boundary conditions which influence the local temperatures and consequently the mechanical behavior. The model, which suggests a one dimensional polycrystal or an imperfect crystal, is shown to slip intermittently in a manner characteristic of ductile solids. Criteria for intermittent slip are derived which are generally compatible with experience. Selected experiments are cited which confirm predictions of the analysis.

INTRODUCTION

This report is one of a series describing exploratory studies of the mechanical behavior of materials which have temperature dependent properties. It is concerned with the behavior of a simple model of a composite material. The earlier studies of homogeneous models showed that when temperature effects are taken into account, the heat produced by the irreversible work of deformation can have a strong influence on the results of an experiment. The extent of this influence depends on the thermal boundary conditions imposed on the model as well as the stress program. Whereas the thermal boundary conditions applicable to the homogeneous model depended on the conditions at the exterior surfaces, in a composite model they depend also on the size and state of dispersion of the components.

The model under consideration is a laminated slab in which uniform, homogeneous, incompressible, viscous layers alternate with uniform perfectly elastic layers. In the absence of temperature effects, the shear behavior of this composite depends only on the properties and relative volumes of the two constituents. When temperature effects are taken into account, the layer thicknesses can be important. At low volume fractions of viscous material, the model suggests a one dimensional polycrystal or imperfect single crystal.

The analysis shows how intermittent slip processes, similar to those characteristic of ductile deformations in real solids, can occur in this model. Criteria for intermittent slip are developed which are found to be compatible with experience. A key parameter in the analysis is the temperature coefficient of viscosity which is related to the energy of activation for the flow process.

As a result, this continuum analysis has a point of contact with the atomic scale physics of solids. Several published experimental studies are cited to demonstrate the plausibility of the model.

GENERAL DISCUSSION OF THE ANALYSIS

If the number of layers of elastic or viscous material in the laminated model is N, the thickness of each elastic layer is ℓ_E, and the thickness of each viscous layer is ℓ_v, the total thickness of the slab L, is given by

$$L = N(\ell_E + \ell_v) \tag{1}$$

The volume fraction of viscous material V is given by

$$V = \frac{\ell_v}{\ell_E + \ell_v} = \frac{N\ell_v}{L} \tag{2}$$

At constant rate of shear deformation, R, the development of stress, σ, in this model can be described by the usual expression for a Maxwell model

$$\dot{\sigma} = G'R\left(1 - \frac{\sigma}{\eta' R}\right) \tag{3}$$

in which $\dot{\sigma}$ is the time derivative of the local stress, and η', the apparent viscosity, is related to the viscosity, η, in a layer by

$$\eta' = \frac{\eta}{V} \tag{4}$$

and G', the apparent elastic modulus, is related to the modulus in an elastic layer, G, by

$$G' = \frac{G}{1-\nu} \tag{5}$$

When G' and η' are constant, as they would be in the isothermal case, Eq. (3) can be integrated to give the familiar form

$$\sigma = \eta' R \left(1 - e^{-(G'/\eta')t}\right) \tag{6}$$

When G' or η' depend on the temperature, and therefore on time, the integration cannot be performed so easily.

Following[1], let us assume that the viscosity depends on temperature T according to

$$\eta = \eta_o e^{-a(T-T_o)} \tag{7}$$

in which η_o is the viscosity at the reference temperature T_o. Then Eq. (3) takes the form

$$\dot{\sigma} = G'R\left(1 - \frac{\sigma}{\eta' R} e^{a(T-T_o)}\right) \tag{8}$$

The temperature, as a function of time and position, depends on the solution of the heat conduction equation for the viscous layer

$$\frac{\sigma^2}{\eta_o} e^{a(T-T_o)} = c\frac{\partial T}{\partial t} - K\frac{\partial^2 T}{\partial y^2} \tag{9}$$

in which c and k are, respectively, the temperature independent heat capacity and thermal conductivity of the material and y is the space coordinate through the layer.

The solution has been discussed in detail elsewhere[1], and analog solutions of Eq. (8) determined[2]. A result from Ref. 1 that is useful here is that for particular thermal boundary conditions, there is a stress, σ_c, at which thermal instability will occur, and the local effective viscosity will drop abruptly. The criterion for stability is the value of the number

$$\mathcal{G} = \frac{a\sigma_c^2 \ell_t^2}{k\eta_o} \tag{10}$$

so that the critical stress depends on the thickness of the viscous layer.

THE THIN VISCOUS LAYER CASE

If the volume fraction of viscous material, V, is very low, the apparent viscosity of the composite, η', will be very much higher than η (Eq. (4)) for low stresses. The viscous process can then be inconspicuous as long as the flow is stable. When instability occurs, local slip and a drop in the local flow stress will be initiated. The subsequent events can take several courses.

As a simplified example, consider, from a quasi-static point of view, a constant rate of deformation shear experiment with the laminated model. The time required to reach the critical stress, σ_c, would be

$$t_o = \frac{L\sigma_c}{GR} \tag{11}$$

After this time, the stress decays abruptly, and the heating subsides. A time representative of that required for the viscous layer to cool is

$$t_c \simeq \frac{c \ell_v^2}{k} \tag{12}$$

A similarity parameter can now be defined as

$$S = \frac{t_o}{t_c} = \frac{L k}{G c \ell_v^2} \cdot \frac{\sigma_c}{R} \tag{13}$$

When S $\gg$ 1, the initial temperature conditions are essentially restored before the stress again reaches a critical value. Then the slip events can be expected to be periodic with a frequency governed by t_o. This type of intermittent slip is associated with the deformation ductile materials. If the viscous layer is thought of as an imperfect region in a crystal, the thermal experience associated with the slip process could heal the imperfection so that the next slip event would occur on a different one of the many thin viscous layers.

However, if S $\ll$ 1, the viscous layer will not have time to cool, and the slip will continue with no stress build up. Since the slip layer is considered to be thin, the action is localized, and this situation can be somewhat similar to brittle fracture, as pointed out by Zener[3]. Thus, the number S can be considered to be a ductility parameter. It is higher at low strain rates and for high thermal conductivity materials. These inferences are generally compatible with experience.

Several details have been omitted from the discussion above. One of these relates to the dynamics of the events following the initiation of slip. When local slip occurs in the essentially elastic continuum in equilibrium at the stress, σ_c , a local stress zero is introduced. This propagates a stress relief wave outward to the boundaries. There the wave is reflected back to its origin. The transit time, which is the duration of the zero stress at the slip, is of the order

$$t_o' = L(\rho/G)^{1/2} \tag{14}$$

in which ρ is the temperature independent density of the material, and $(G/\rho)^{1/2}$ is the acoustic velocity in the slab. This leads to a new similarity parameter

$$S' = \frac{t_o'}{t_c} = \frac{L\kappa\phi}{c\,\ell_1}\cdot\left(\frac{\rho}{G}\right)^{1/2} \tag{15}$$

which would also be related to ductility.

Another detail that must be considered relates to the energy available to promote the slip catastrophe. In the model itself this would be given by the stored elastic energy.

$$W_s = \frac{\sigma_c^2 L}{2G} \tag{16}$$

In an experiment, energy stored in the testing machine would also have to be taken into account. The ultimate temperature rise, and therefore the severity of the thermal catastrophe, would depend on the gross energy available.

DISCUSSION OF RESULTS

The analysis of the behavior of the model laminated slab given above is by no means a complete answer to the problems of ductility and brittleness. However, it does introduce an approach that could be fruitful. If it has any relevance to the behavior of real materials, the elementary slip processes should be accompanied by the generation of local high temperatures. This is observed in recent studies of Erdmann and Jahoda[4] at very low temperatures.

Furthermore, in a study of the general heating associated with the plastic deformation of aluminum, Dillon[5] found that the signals from fine thermocouples attached to a deforming rod were very "noisy". Since he was only interested in the general heating, Dillon used a small capacitor to suppress this noise. It is possible that the spikes in the thermocouple voltages which he filtered out arose from individual slip events.

Acoustical disturbances are also often produced during the plastic deformation of metals. In the discussion of the simple model given above these would be expected as a result of the propagation of stress relief waves from the slip sites.

Certain somewhat quantitative statements can also be made on the basis of the above analysis. Consider the parameter S' Eq. (15). For a one cm. length of aluminum, t_o' has the value 2×10^{-6} sec. When $t_c = 2 \times 10^{-6}$, $l_v \simeq 2.2 \times 10^{-3}$ cm. In other words, in ductile aluminum the effective size of the viscous layer must be less than 2.2×10^{-3} cm. This size is large compared with typical regions of crystal imperfections in aluminum so that the ductility would be expected.

In a real material the regions of crystal imperfection are very small but they may move about in a stress field, occasionally forming "clusters". The effective size, for thermal relaxation, of a cluster would be much larger than an isolated region. The formation of a cluster could then precipitate the slip instability.

ACKNOWLEDGEMENTS

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Thermistor Analogs for Model Viscous and Viscoelastic Systems

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Synopsis

The physical and mathematical similarity of the resistance–temperature relation in thermistors to the viscosity–temperature relation in typical liquids makes it possible to simulate the essential nonlinearity of the flow of model Newtonian liquids by the use of an electric analog. Simple measurements of current and voltage take the place of detailed computation. Circuits containing a thermistor and capacitor provide a model for nonlinear viscoelastic materials. A transmission line containing thermistors would be applicable where inertial terms are important and both the time and space variation of the stress would be significant. The behavior of simple analogs described here agrees with available numerical computations. In addition, it duplicates experimentally observed temperature effects and apparent departures from Newtonian behavior in liquids as well as necking, yield, fracture, creep, strain hardening, and stick-slip effects in solids. This electric analog literally simulates the model mechanical system. In contrast, the widely discussed simulation of linear viscoelasticity by electric networks is figurative. That is, there, the well-developed mathematics of linear circuits was applied to the mechanical system. The new simulation is effective over wide ranges of stress, strain, and time even with a single thermistor.

INTRODUCTION

This report describes part of a general exploratory study of the mechanical behavior of materials with temperature dependent properties. With such materials, the heat produced in a mechanical experiment can have a strong influence on its outcome. Calculations of the magnitude of this influence in several idealized but representative experiments were described in earlier reports.[1,2] The analyses were straightforward but required the use of high speed computing equipment. It will be shown here that results similar to those obtained from the computation can be obtained by the use of electric analogs which contain temperature-dependent electrical resistance elements (for example, thermistors). The analog method is fast and inexpensive and may be better adapted to the solution of some problems than the computation.

The precisely analogous properties of electrical networks and idealized mechanical systems have been reviewed by Alfrey.[3] When the mechanical systems are linear the electrical analogs are used figuratively. That is, in reality, the well-established mathematical methods applicable to linear circuits are directly applied to the mechanical problem. In nonlinear systems no completely adequate mathematical methods have been developed. Therefore, in the present work, the literal application of the analogs has been undertaken.

Fortunately, the temperature dependence of the electrical resistance of thermistors is identical in form with an acceptable expression for the temperature dependence of the viscosity of condensed phase materials.[4] In the thermistor and in the simulated materials, the flow process involves biased diffusion and has a characteristic energy of activation. The relationship between this energy of activation and the temperature coefficients provides a bridge between the present, essentially phenomenological, discussion and the elegant contemporary atomic scale models of the deformation process that have been proposed.

For this discussion, it is convenient to divide one-dimensional mechanical experiments with homogeneous isotropic incompressible materials into two major classes, although acknowledging the existence of a transition region. Slow or quasi-static experiments are in one class. Here the time of the experiment is very long compared with the time required for stress equilibrium to be established in the test piece. Consequently, while the stress may be time-dependent, its space variation is negligible. In this case, inertial terms in the energy and momentum equations are also negligible. Fast experiments constitute the second major class. Here both the space and time variations of the stress are of consequence and inertial terms are not negligible. Notice that the size of the sample affects the equilibration time as much as the velocity of propagation.

A transmission line containing thermistors as well as inductors and capacitors is the general simulation. The space dependence of the stress is simulated by the values of the voltage along the elements of the line. The currents in the contiguous loops simulate the average velocity gradients in contiguous elements of the test piece. Notice also that heterogeneities in the material can be simulated by variation of the transmission line parameters.

If in an experiment the inertial terms are negligible, the voltage will show negligible variation from loop to loop. That is the transmission line is not necessary in the quasi-static case.

The thermistor analog measurements described in detail below involve simple circuits and apply to quasi-static experiments. These are simpler conceptually and easier to arrange than studies of transmission lines. In addition, the results can be compared with the computations of refs. 1 and 2 in which only quasi-static cases were considered.

It may be noted in passing that the responses of materials to sinusoidal strain programs that are frequently observed in laboratory studies are usually treated as quasi-static. That is, inertial terms are neglected. The apparent dynamic feature of these experiments arises from the fact that viscoelastic systems have characteristic natural periods associated with the ratio of the viscosity to the elastic modulus of the components of the element.[3] In these cases, the amplitude and phase of the periodic stress response depend on the relationship between the period of the strain program and the natural period.

The detailed mathematics of linear systems are adequately discussed in the literature. The detailed mathematics of the nonlinear systems have been discussed in refs. 1 and 2 and will not be elaborated upon here.

Before concluding these introductory remarks, attention is drawn to one facet of the results. That is, simple, single circuits containing thermistors show behavior analogous to yield, plastic flow, strain hardening, and fracture as well as stick-slip effects which are typical in the deformation of ductile metals. In addition, effects of changes in the ambient temperature and thermal boundary conditions on mechanical behavior can also be explicitly simulated. Alternative phenomenological descriptions of these common types of mechanical response are generally more complicated.[4]

THE VISCOUS ELEMENT

In the traditional viscous element, or dashpot, used in mechanical models, the stress σ and strain rate $\dot{\epsilon}$ are related by Newton's law.

$$\sigma = \eta\dot{\epsilon} \tag{1}$$

in which η is the viscosity of the element. This is analogous to the behavior of an electrical resistance element in which the voltage V and the current I are related by Ohm's law

$$V = RI \tag{2}$$

in which R is the resistance of the element. In the temperature-dependent viscous element, the relationship between the observed strain rate and the stress is nonlinear and more complicated than eq. (1). The general equations were developed in detail in ref. 1 but will be reviewed briefly here.

Consider the plane shear of an infinite slab of viscous material of unit thickness in the y direction (Couette flow). The displacement rate of one boundary relative to the other is the integral of the velocity gradient $\partial u/\partial y$, through the thickness. That is,

$$\dot{\epsilon} = \int_0^1 (\partial u/\partial y)dy. \tag{3}$$

In the quasi-static case, the shearing stress is independent of y and is assumed to be related to the local values of the velocity gradient and viscosity by the usual viscosity law

$$\sigma = \eta(\partial u/\partial y). \tag{4}$$

Using for the temperature dependence of the viscosity

$$\eta = \eta_0 e^{(E_\eta/B)(1/T-1/T_0)} \tag{5}$$

in which η is the viscosity at the absolute temperature T, η_0 is the viscosity at a reference temperature T_0, E_η is the energy of activation for the flow process, and B is the Boltzmann constant, eq. (3) can be rewritten

$$\dot{\epsilon} = (\sigma/\eta_0) \int_0^1 e^{(E_\eta/B)(1/T_0-1/T)}\,dy. \tag{6}$$

Clearly, when $T = T_0$ (the isothermal case) eqs. (6) and (3) are identical. However, if the local temperatures are not constant or are different from the reference temperature the equations are not the same.

The numerical procedures in ref. 1, in effect, provided values of the time-dependent integral in eq. (6) subject to the mechanical work input and the thermal boundary conditions imposed on the slab. It was shown that for the case of cooled walls the value of the integral remains close to unity when a certain nondimensional parameter has small values. This number is

$$\mathcal{G} = a\sigma^2 l^2/k\eta_c \tag{7}$$

in which l is the thickness of the slab, k is the thermal conductivity of the material, and a is the temperature coefficient of viscosity which is approximately equal to E_η/BT_0^2 in the notation used above. A similar argument can be applied to the relationship between the applied voltage and the current in a thermistor of comparable geometry for which

$$R = R_0 e^{(E_r/R)(1/T - 1/T_0)} \tag{8}$$

where E_r is the energy of activation for the electrical conduction process. Then the value of the integral will be given by

$$I = (V/R_0) \int_0^1 e^{(E_r/R)(1/T_0 - 1/T)}\, dy. \tag{9}$$

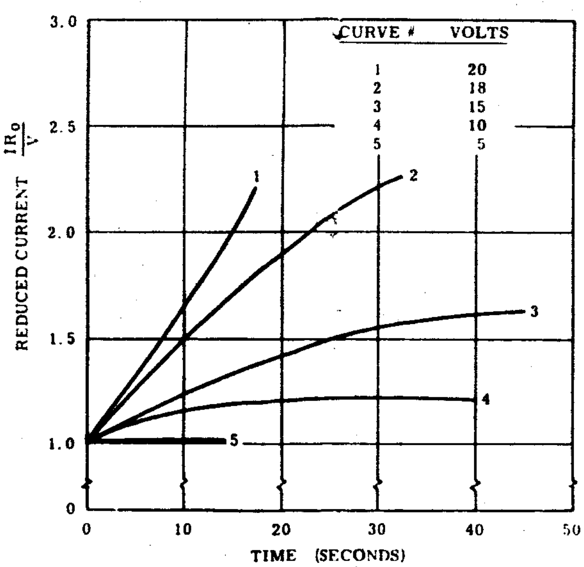

Fig. 1. The reduced time-dependent current through a thermistor at various constant voltages. Analogous to the reduced time-dependent shear rate of viscous model at various constant stresses.

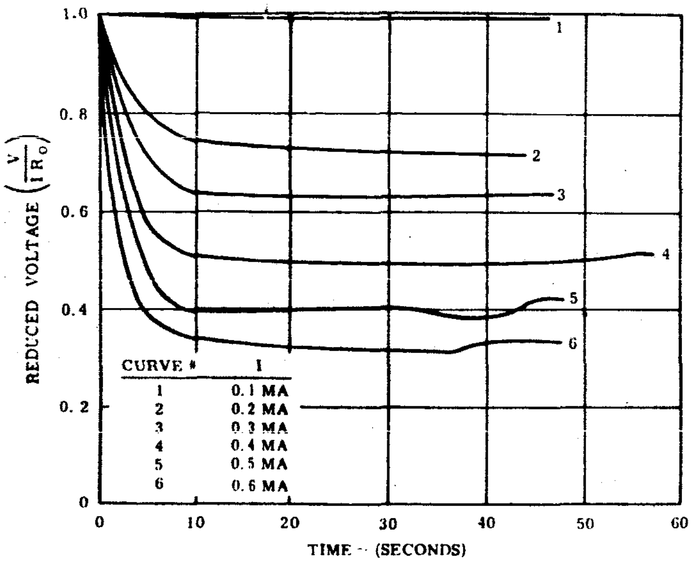

Fig. 2. The reduced time-dependent voltage across a thermistor at various constant currents. Analogous to the reduced time-dependent stress on a viscous model at various constant strain rates.

The departure of the value of the integral from unity will depend on

$$\mathcal{G} = aV^2l^2/kR_0. \tag{10}$$

The integral in eq. (9) is identical with that in eq. (6). The use of measured values of current and voltage to evaluate the integral in eq. (9) is the essential feature of the thermistor analog method.

Observed time-dependent values of IR_0/V for various constant values of V for a typical thermistor are plotted in Figure 1. This plot is similar to Figure 2 of ref. 1 in which the computed time-dependent values of $\dot{\epsilon}\eta_0/\sigma$ are plotted for various constant values of σ. In both cases, the instability of the system at high stresses is clearly shown.

Figure 2 is a plot of the observed values of V/I_0R for various constant values of I and for the same thermistor used for the constant voltage experiments of Figure 1. This plot must be compared with Figure 7 of ref. 1.

The nominal values of the resistance of the thermistor used in these two experiments was 100,000 ohms at room temperature.

The nominal thermal relaxation time was 2 sec and the nominal temperature coefficient of resistance was 4.5%/°C.

THE MAXWELL MODEL (CONSTANT STRAIN RATE)

The Maxwell model is a viscoelastic element in which a viscous component and an elastic component are connected in series. Its electric analog is a resistance and capacitor in parallel.[3] When a constant current I is passed through a linear circuit of this type starting with an uncharged condenser, the voltage rises from zero to the value IR in a time determined by the product of the values of resistance and capacity. In the nonlinear analog, the value of R can be time-dependent, decreasing as the temperature rises according to eq. (9). This can change the time constant of the circuit and produce a voltage maximum analogous to what would be called yield and plastic flow in the nonlinear mechanical model.

Plots of typical time-dependent voltages for various constant currents are presented in Figure 3. When the current is low, the behavior of the model is essentially linear. As the current is increased, a peak in the voltage–time curve appears which is the manifestation of the nonlinearity. At higher currents, damped oscilla-

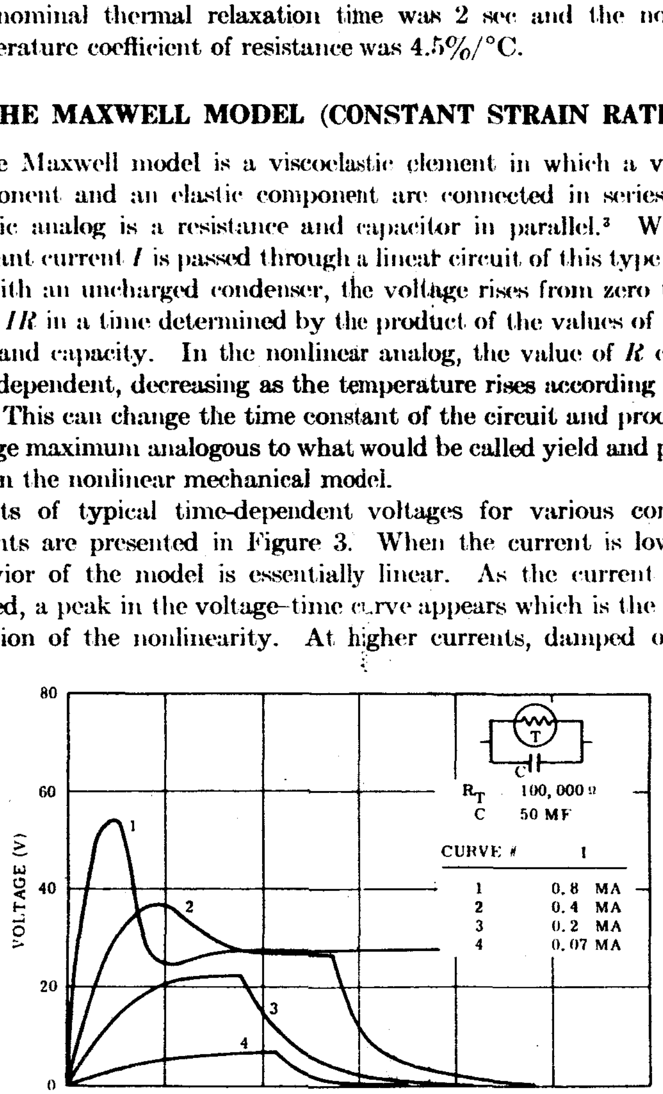

Fig. 3. The time-dependent voltage across a parallel connected thermistor and capacitor at various constant currents. Analogous to the stress on a Maxwell model at various constant strain rates.

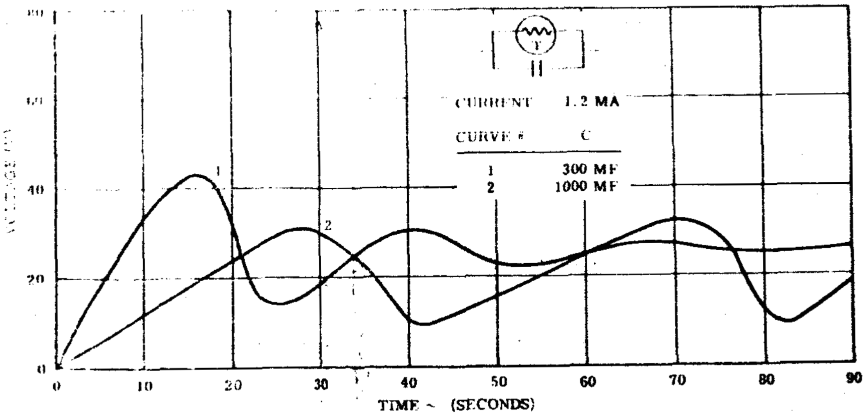

Fig. 4. Same as Figure 3 but showing sustained oscillations analogous to stick-slip in the mechanical model.

tions appear in the voltage time curves because of the augmentation of the current through the thermistor by the discharge from the capacitor.

As shown in Figure 4, in selected circuits, the oscillations are maintained because the thermal recovery time of the thermistor is less than the charge time of the capacitor. This phenomenon is analogous to stick-slip effects in the mechanical model. If the stored energy in the capacitor is high enough, the discharge current can destroy (vaporize) the thermistor producing the effect analogous to fracture.

The results presented in Figures 3 and 4 were obtained with circuits containing thermistors similar to those used to simulate the viscous element described earlier. The values of the current and capacity are given in the figures. Discontinuities in the curves occur when the current is turned off. The subsequent events are analogous to stress relaxation. It may be seen that this process depends not only on the characteristics of the circuit but also on the current history because of its effect on the value of the temperature, and therefore resistance, of the thermistor.

The responses of the Maxwell model or its electric analog to a step function in stress or voltage respectively, involve very high instantaneous strain rates or currents. The quasi-static assumption made here is less appropriate for this problem. The transmission line analog is more suitable for nonlinear creep and creep recovery experiments.

THE KELVIN-VOIGT MODEL (CONSTANT STRAIN RATE)

The Kelvin-Voigt model is a viscoelastic element in which a viscous component and an elastic component are in parallel. Its electrical analog is a resistance and capacitor in series. When a constant current is passed through a linear circuit of this type, starting with an uncharged condenser, the initial value of the voltage is IR and the voltage increases with time at a rate given by the ratio of the current to the capacity. In the nonlinear analog, the voltage

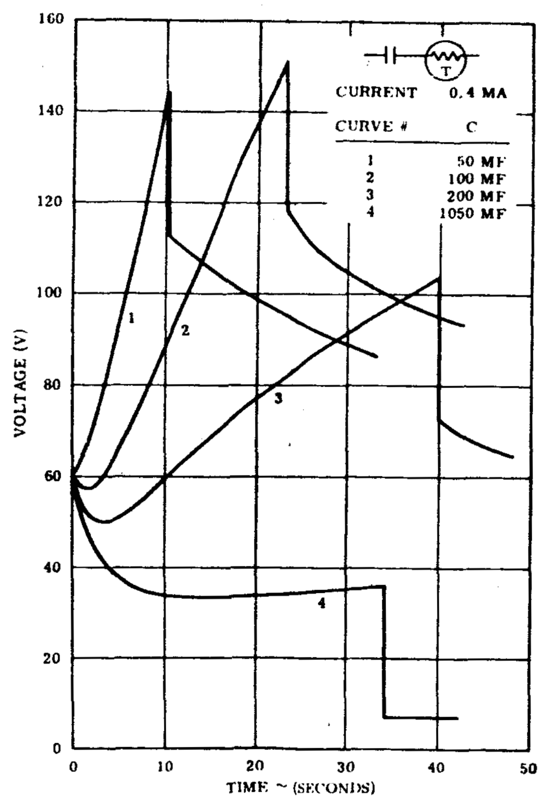

Fig. 5. Same as Figure 3, but for a Voigt model.

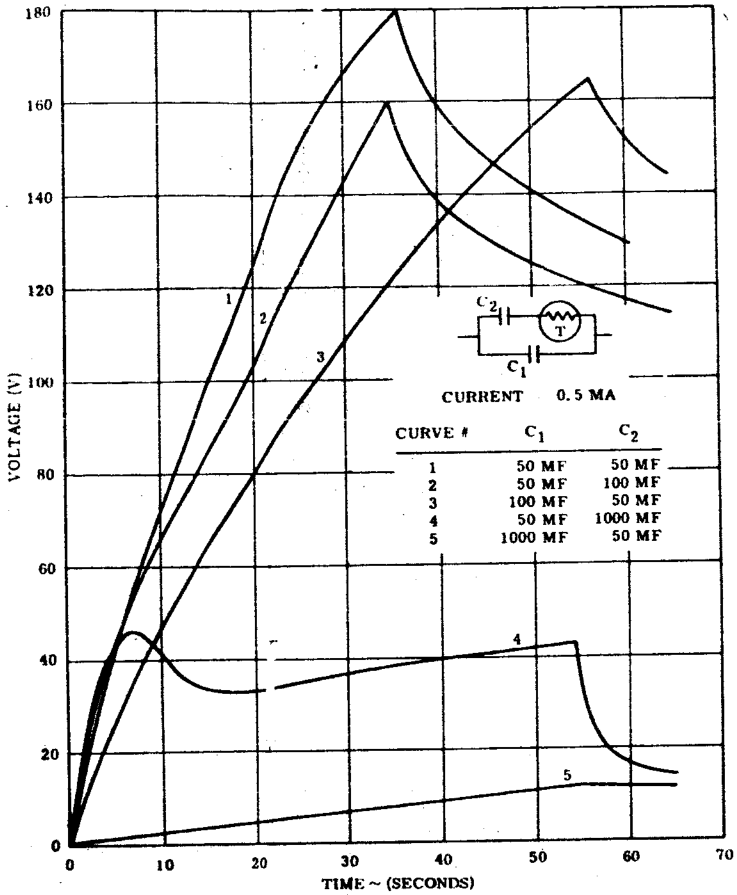

Fig. 6. Same as Figure 3, but for a three-component model.

can drop as the thermistor warms up but the voltage will then rise again as the charge accumulates on the condenser. This type of behavior is analogous to what would be called yield with strain hardening in a mechanical model.

Plots of typical observed time-dependent voltages at constant current for various values of the capacity are presented in Figure 5. When the capacity is low, the response is nearly linear because the voltage build up on the condenser occurs before much energy is dissipated in the thermistor. With higher values of the capacity, the energy dissipation in the thermistor goes up. Yield is more conspicuous and failure can occur when the temperature in the thermistor rises. As before, the stress relaxation phenomenon is also exhibited and it can be expected to depend on the over-all strain history. This group of curves should be compared with Figure 8 of ref. 2.

A THREE-COMPONENT MODEL

While viscous and elastic components and their electrical analogs can be permuted and combined almost endlessly, we will terminate the discussion of configurations here with a description of three-component models consisting, in the mechanical case, of an elastic component in series with a parallel array of elastic and viscous components. The electrical analog is shown diagrammatically in Figure 6 together with some examples of the observed time-dependent voltages at constant current.

Here, depending on the current level and on the relative values of the resistance and capacities, a wide variety of curves analogous to stress strain curves can be generated. Some of these simulate linear elasticity. Others appear to show strain- or stress-dependent moduli. Some show yield and strain hardening. The stress relaxation following the cessation of current flow is also quite different in each of the experiments.

TEMPERATURE EFFECTS

Not only are other circuit configurations feasible but other types of analog experiments are possible. For example, quasi-static responses of the models at different temperatures can be considered. The effect of temperature on the electric analog is for the most part equivalent to changing the initial value of the thermistor resistance R_0. This can be accomplished by replacing the thermistor with one of a different nominal resistance or by literally changing its initial temperature. The consequences of such changes can readily be anticipated and need not be elaborated upon here, since the selection of the thermistor was in the first place quite arbitrary.

Insofar as size effects are due to changes in the thermal economy, these can also be directly simulated by using different size thermistors (a wide range of sizes is available) or by changing the thermal boundary conditions on a single thermistor by forced circulation of air or immersion in a still or flowing liquid.

While, in general, raising the temperature has the effect of reducing the time constant of the thermistor circuit, the effects of time and temperature do not have to be separable, so that this approach does not suggest, directly, the idea of time–temperature superposition.[5] Notice that in the above model the temperature effects are localized in the viscous component. In real materials, the viscosity component will have a stronger dependence on temperature than the elastic component but the latter is not likely to be zero.

NECKING

The analog of a series arrangement of similar nonlinear viscous elements is a parallel array of similar thermistors. This circuit does not strictly fit with the group of analogs described above. However, the behavior of the array so nicely simulates the common phenomenon of necking during deformation that it will be discussed here.

When the sum of the currents through the thermistors is low, the separate currents will be essentially equal. Now, as the total current is increased one of the thermistors will inevitably heat up more than the others. This thermistor will then have a lower resistance and it will pass more and more of the total current. Since the current is the analog of strain rate, the concentration of the current is analogous to necking.

CONCLUSION

Selected analog experiments showing the type of nonlinear effects which can be expected in the behavior of mechanical models with temperature-dependent properties are described. Only quasi-static one-dimensional experiments with incompressible homogeneous isotropic materials are discussed in detail. Possible methods for dealing with the more general mechanical problems involving dynamic experiments with heterogeneous materials are suggested. It is also possible that the use of more complicated electrical networks would permit the extension of the capability of the analog to the two- and three-dimensional problems.

In the linear analog simulation,[3] attempts are made to match intricate observed responses of materials by the use of distributions of linear elements. While it will always be possible to match a particular response by an astute choice of a spectrum of linear elements, the reality of many of the assignments of relaxation times has yet to be demonstrated. When nonlinear elements are admitted, the complexity of the required spectrum may be much reduced.

In any event, to the extent that this approach leads to more accurate descriptions of mechanical behavior, it may aid structural design and improve the correlations of behavior with atomic scale structure.

The study program described above has been supported in part by the Office of Advanced Research and Technology of the National Aeronautics and Space Administration under Contracts NASw708 and 1190 monitored by Messrs. Howard Wolko and Melvin Rosché. The author is also indebted to his colleagues, James Wolfe and John Pollock for assistance with the experiments.

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ABSTRACT

An electric transmission line analog for studies of the dynamic plane shear of an incompressible viscoelastic material which has a temperature dependent viscosity is described. Thermistors are used to simulate the temperature dependent viscosity. Criteria are established for the similarity of the line and the material. Experiments analogous to constant rate of deformation studies of an elastic material and a material with a single viscoelastic relaxation time are described. More detailed experiments analogous to the deformation of a viscous material at a constant rate of deformation and at constant stress are also described. These show phenomena analogous to necking and fracture.

THERMISTOR ANALOG STUDY OF DYNAMIC SHEAR IN MODEL VISCOELASTIC MATERIALS

I. INTRODUCTION

This report is one of a series describing exploratory studies of the mechanical behavior of materials which have temperature dependent properties. With such materials, the heat produced by the irreversible work or deformation can have a strong influence on the outcome of an experiment. The general problem considered here relates to the dynamic, plane shear of an ideal, incompressible viscoelastic material. The approach that is used involves an electric transmission line analog containing temperature sensitive resistors (thermistors). This study of the dynamic shear problem follows naturally from earlier work in which a thermistor analog was applied to the study of the quasi-static shear of model viscous and viscoelastic materials (1).* Before that, numerical solutions of the same problems had been reported (2). The results of the quasi-static analog study were similar to those of the numerical study but simpler to obtain.

Except for the use of shunt thermistors to simulate the temperature dependent viscosity in the model material, instead of ordinary resistors, the transmission line analog is similar to those discussed by Mason (3) and others. That is, line inductors and shunt capacitors are used, respectively, to simulate inertial effects and elastic effects in the material. Since the temperature dependence of viscosity is very much stronger than the temperature dependence of density or elastic modulus, it is reasonable to neglect the latter in this exploratory study. In the discussion below, some general features of the elastic and viscoelastic systems are considered. Then the systems with

*Numbers in parentheses refer to the Bibliography.

viscosity alone are discussed in greater detail. No real materials have viscosity alone. However, in selected organic polymers and their solutions, the effects of elasticity and compressibility are subordinate to those of viscosity.

From an engineering point of view, the general study may suggest rational bases for raising design allowable stresses for short duration loads. From a material development point of view, relationships between atomic scale structure and mechanical behavior in terms of an energy of activation are suggested.

II. GENERAL DISCUSSION OF THE THEORY

A. Wave Equations

The usefulness of the electric analog in mechanical studies depends on the similarity of the equations describing the voltage and current history in the transmission line and the equations describing the stress and strain rate history in the mechanical model. While the work described here is experimental, it is done in a theoretical context which is reviewed briefly. In the transmission line, shown schematically in Figure 1, the current (I) and the voltage (V) depend upon both the time (t) and the distance (y) from the end of the line. The equations are:

$$\frac{\partial V}{\partial y} = L \frac{\partial I}{\partial t} \qquad \text{1a}$$

$$\frac{\partial I}{\partial y} = C \frac{\partial V}{\partial t} + \frac{V}{R} \qquad \text{1b}$$

in which R, L, and C are, respectively, the shunt resistance, inductance, and capacity each per unit length. These equations for an ideal line (no resistance in the inductor) are exactly analogous to Newton's second law of motion and the general stress-strain relation as written for an ideal Maxwell solid

$$\frac{\partial \sigma}{\partial y} = \rho \frac{\partial u}{\partial t} \qquad \text{2a}$$

$$\frac{\partial u}{\partial y} = \frac{1}{G} \frac{\partial \sigma}{\partial t} + \frac{\sigma}{\eta} \qquad \text{2b}$$

in which σ is the stress, u is the velocity, ρ is the density, G is the appropriate elastic modulus, and η is the viscosity. (See for example Kolsky Reference 4.)

These equations may be combined within their own system to obtain wave equations applicable to various types of materials. For example, if η is infinite, the material is perfectly elastic and the wave equation becomes

$$\frac{\partial^2 u}{\partial y^2} = \frac{\rho}{G} \frac{\partial^2 u}{\partial t^2} \qquad \text{3a}$$

In the electrical analogy, R is set equal to ∞ so that

$$\frac{\partial^2 I}{\partial y^2} = L C \frac{\partial^2 I}{\partial t^2} \qquad \text{3b}$$

is the wave equation for a perfectly lossless transmission line. If $\frac{1}{G}$ is zero, one may obtain the Navier Stokes equation (5) for a perfectly viscous material:

$$\frac{\partial u}{\partial t} = \frac{1}{\rho} \frac{\partial}{\partial y} \left(\eta \frac{\partial u}{\partial y} \right) \qquad \text{4a}$$

By analogy, one sets C = 0 and obtains the expression

$$\frac{\partial I}{\partial t} = \frac{1}{L} \frac{\partial}{\partial y} \left(R \frac{\partial I}{\partial y} \right). \qquad \text{4b}$$

For a viscoelastic material, in which the coefficients of equations 2 are constant, the damped wave equation is applicable.

$$\frac{\partial^2 u}{\partial y^2} = \frac{\rho}{G} \frac{\partial^2 u}{\partial t^2} + \frac{\rho}{\eta} \frac{\partial u}{\partial t} \qquad \text{5a}$$

By similar algebraic treatment the electrical analogy is

$$\frac{\partial^2 I}{\partial y^2} = LC \frac{\partial^2 I}{\partial t^2} + \frac{L}{R} \frac{\partial I}{\partial t} \qquad \text{5b}$$

which is the usual transmission line equation, if the inductance is lossless. Analytical solutions to equations 5 have been discussed numerous times and electrical analogies have been applied to their study. (See references 5 and 6).

This report concerns the effects produced within a viscous material when the viscosity depends on temperature. In this case, the analysis is very difficult, if not impossible. However, equation 4b shows that if R would have the same variation with distance as η, the analogy would still be valid. The viscosity has been found to vary with the temperature according to the equation

$$\eta = \eta_o \, e^{+\frac{W}{B}\left(\frac{1}{T} - \frac{1}{T_o}\right)} \qquad \text{6a}$$

For the thermistors the resistance varies according to the relation

$$R = R_o \, e^{+\frac{W}{B}\left(\frac{1}{T} - \frac{1}{T_o}\right)} \qquad \text{6b}$$

in which the zero subscripts refer to an initial or reference condition, W is the approximate energy of activation and B is Boltzmann's constant.

The temperatures of the materials and the thermistors are determined by the solutions of the energy equations

$$\eta_o \left(\frac{du}{dy}\right)^2 e^{\frac{W}{B}\left(\frac{1}{T} - \frac{1}{T_o}\right)} = C_p \frac{\partial T}{\partial t} - K \nabla^2 T \qquad 7a$$

$$R_o \left(\frac{dI}{dy}\right)^2 e^{\frac{W}{B}\left(\frac{1}{T} - \frac{1}{T_o}\right)} = C_p \frac{\partial T}{\partial t} - K \nabla^2 T \qquad 7b$$

in which the left hand member is the mechanical or electrical power dissipated, $C_p \frac{\partial T}{\partial t}$ is the locally stored power and $K \nabla^2 T$ is the power loss by heat conduction. The symbols c_p and k are the volumetric heat capacity and thermal conductivity respectively. The details of these relationships have been discussed for the quasi-static cases in reference (1) and (2). It may be said here, in summary, that the thermistor resistance automatically varies with time in a manner completely analogous to the variations of viscosity dictated by equations 4 and 6.

B. Similarity Parameters

No analytical discussion of the transmission line equations subject to equations 6b and 7b will be given. However, two similarity parameters for the deformation process will be developed which are useful for characterizing experiments.

To obtain the first parameter notice that the diffusion equation (e.g. equation 4) defines a characteristic time

$$t_D = \frac{\rho \ell^2}{\eta} \qquad 8$$

in which ℓ is the thickness of the material and which relates to the duration of the inertial transient in the model. There is another characteristic time relating to the heating process derivable from the adiabatic form of equation 7A

$$t_h = \frac{C_p}{a \eta_o u_o^2} \qquad 9$$

in which a is the temperature coefficient of viscosity, which is related to the energy of activation of flow, and u_o is the velocity of boundary motion. The value of the ratio

$$D = \frac{t_D}{t_h} = \frac{a \rho \eta_o^2 \ell^2}{C_p} \qquad 10$$

or in the analog equations

$$D = \frac{a L I^2 \ell^2}{C_p} \qquad 11$$

indicates the relative duration of the inertial transient and the thermal transient. When D is small, the inertial transient is relatively uninfluenced by the slower heating effects and the quasi-static treatment of the thermal effects given in references 1 and 2 is appropriate. When D is large, the interaction of the thermal and inertial effects is significant.

One more characteristic time arises from the elastic equation.

$$t_E = \sqrt{\rho/E}\, \ell \qquad 12$$

The value of the ratio

$$F = \frac{t_E}{t_D} = \frac{\eta_o}{\sqrt{\rho E}\, \ell} \qquad 13$$

or in the analog

$$F = \sqrt{\frac{C}{L}}\ \frac{R_o}{\ell} \qquad 14$$

characterizes the line and indicates the relative effect of the elastic wave and the diffusion wave. At low values of ℓ the diffusion process travels faster.

III. DESCRIPTION OF APPARATUS

The ten loop transmission line shown schematically in Figure 1 was assembled from commercial components. Inductors, capacitors, thermistors and resistors were selected to make it possible to do experiments at conveniently measurable voltages, currents and times. Non-idealities are introduced by the resistance associated with the inductance and by the fixed ammeter resistor in series with each thermistor. These are minimized to prevent interference with the essential features of the observations.

The inductors are each iron core filter chokes with nominal inductances of 8 henrys at 300 milliamperes. Their resistance is 80 ohms. The capacitors are metallized Mylar and have nominal values of 2.0 MF. The thermistors are rated 10,000 ohms at 25°C and have a temperature coefficient of 5.5% per degree centigrade. They each weigh 0.100 gms and they are made of a material with a heat capacity of about 0.60 joules per gram per degree centigrade. Thus their heat capacity is about 0.06 joules per degree. The ammeter resistances are 100 ohms.

The thermistors were disc type 0.22" diameter and 0.036" thick. These were fitted with thin brass electrodes and stacked using 0.001" Mylar film spacers so that they were insulated from one another electrically but in thermal contact. The entire stack was thermally lagged except on one face which was in contact with an aluminum heat sink. This configuration simulates an infinite slab having a heat sink at one of its boundaries and insulated at the other. This stacking feature is not important for the short time experiments reported here but is mentioned to complete the description of the apparatus.

IV. EXPERIMENTAL RESULTS

A. Elastic Wave Propagation

The first group of experiments to be described were performed with the line shown in Figure 1, but without the shunt resistances. This LC line simulates a perfectly elastic material. The results confirm predictions. They are given here only to contrast with the results of the viscoelastic and viscous analogs discussed later.

A constant current, Ia, was applied at station 1 at t = 0. Voltage histories at the numbered stations were then recorded. This experiment is analogous to observing the propagation of a stress wave initiated by applying a constant shear velocity to a boundary of a slab of perfectly elastic material. The results for I_a = 5 milliamp are shown in Figure 2. The initial voltage wave height is 15 volts and the observed velocity is 112 loops per second. The calculated wave height, based on the nominal value of the components is $V = I_a\sqrt{\frac{L}{C}}$ = 10 volts. The calculated velocity is $u = \frac{1}{\sqrt{LC}}$ = 250 loops per second. It seems likely that the inductance of the chokes at lower than the rated current, exceeds the nominal. In fact, measurement of the choke at 5 milliamperes showed 13 henrys. Although this variation of L was unexpected, it does provide an opportunity to introduce the effect of density changes during the deformation of materials into the analog.

At each station the voltage rises stepwise with time; the steps becoming less and less distinct as the voltage increases. The experiment is terminated when the voltages go beyond the range of the meters (300 volts).

B. Viscoelastic Wave Propagation

The second group of experiments was performed with constant current applied to line of Figure 1 with both the capacitors and thermistors in place. These showed more complicated effects. The first point to be noted is that the increase in voltage (stress) is arrested. The peak voltage cannot exceed $I_a R_o$ when all of the current goes through the unheated first thermistor. The steady voltage cannot exceed $I_a Ro/10$ when the ten thermistors are unheated and sharing the current equally.

The observed values of the peak voltage are always less than $I_a R_o$ because, in an ideal case, part of the initial current goes to charge the first capacitor. This enables current to reach the second and subsequent loops before the voltage reaches this peak value. The steady voltages only reach $I_a R_o/10$ when current is low and no heating occurs in any of the thermistors. These maximum voltages, which depend on the thermistors, can be called the strength of the line and are analogous to the maximum stresses in the model material which depend on the responses of the viscous elements. One of the rather complicated records is shown in Figure 3. Notice that when the parameter F (equation 14) has the value of 1 (at about loop number 4) a phenomenon occurs which may be interpreted to show the emergence of the elastic wave and its sharp attenuation.

C. Viscous Wave Propagation

The main subject of this study is the interaction between thermal and inertial effects in the viscous model. The preliminary experiments are intended to show how this interaction is likely to fit into the larger more complicated questions relating to viscoelastic behavior. They show, in particular, how the ultimate stress bearing capability of a model viscoelastic material can be limited by its viscosity and how the stress peak due to inertia effects can be limited by the elasticity.

The third group of experiments is a more detailed study of the diffusion of strain and stress in the viscous model. A constant current is applied to the beginning of the line (not containing condensors) and voltage histories at the numbered and lettered stations were recorded. The voltages at the lettered stations indicate the local current; that is, $^{dI}/dy$. When a low current is applied to the beginning of the line, the time ratio D is low and isothermal propagation is represented. The records for I_a = 5 milliamperes (D=.018) are presented in Figure 4. The voltage fails to rise to the value $I_a R_o$ because of the finite voltage rise time in the power supply. However, the steady voltages are $I_a R_o/10$ showing that no significant heating has occurred in the time of the experiment.

At higher values of D (I_a=150 milliamperes, D=16.2, Figure 5) the initial peak is attenuated by the heating effect in the first thermistor. No steady state is shown. However, the tendency for the current to concentrate in the first thermistor at the expense of the others is clear. This type of behavior can be expected whenever high currents are passed through equivalent thermistors in parallel. It is considered to be analogous to necking in the mechanical model.

At even higher currents (200 milliamperes, D=28.8, Figure 6) the decay of the resistance in the first thermistor, due to heating, is after a finite time, catastrophic. This leads to an abrupt decline in the voltage at the first station. This may be analogous to "fracture" in a viscous material. It may be seen in Figures 4 and 5 that at the higher current the height of the voltage peak becomes less than proportional to the current. The post peak voltages are lower for the higher currents. This is analogous to plastic flow in which the stress tends to become independent of the strain rate.

The time to the abrupt decline of voltage would become shorter at even higher currents, but the experiment described above is at the limit of the apparatus.

The non-uniformity of the voltages at the various numbered stations in Figures 4, 5, and 6 are partly due to the voltage drop in the 80 ohm resistance of the inductor and partly due to the dR/dy term in equation 4b when R is a function of y. (It is also a function of t but this will only appear in the equation for the voltage.) Thus,

$$L \frac{\partial I}{\partial t} = R \frac{\partial^2 I}{\partial y^2} + \frac{\partial I}{\partial y} \frac{\partial R}{\partial y} \qquad 15a$$

$$\frac{\partial^2 V}{\partial y^2} = \frac{L}{R} \frac{\partial V}{\partial t} - \frac{LV}{R^2} \frac{\partial R}{\partial t} \qquad 15b$$

D. Constant Voltage Experiments

In the fourth and final group of experiments, various constant voltages, V_a, were applied to the line. As before, the voltage histories at both the numbered and lettered stations were recorded. The results of an experiment at low voltage (V_a=3 volts) are shown in Figure 7. The voltage at station 1 shows a finite rise time and an initial peak. The rise time is short and is likely to indicate the response time of the recorder. The peak is probably a characteristic of the power supply. These curves show the isothermal behavior of the system. The duration of the voltage transient has close to its calculated value of $\frac{L\ \ell^2}{R_o}$=.08 seconds.

At much higher voltages the heating is significant and the resistance of the thermistors changes rapidly. In Figure 8, the records for V_a=150 volts are shown. The temperature of the first thermistor is rising rapidly and approaching catastrophe. The effects are much less pronounced in the thermistors farther down the line. These data suggest that high voltages can be tolerated by the line if the time of application is so short that little heating occurs. They show further how the currents, or in the mechanical model, the deformations can be quite heterogeneous particularly if the duration of the voltage pulse is less than the time constant on the line. Under these conditions the pulse of voltage or current reaching the end of the line can be grossly distorted and attenuated.

V. CONCLUDING REMARKS

Direct comparisons of the results obtained with the analog with the results obtained from mechanical studies on real materials are somewhat premature. This work is more properly considered an introduction to a type of device useful for the study of non-linear mechanical behavior. It establishes criteria for similarity between the mechanical model and the electric analog. These are, in the linear case, the viscoelastic time $\eta/_G$ in the model and RC in the analog; and the new parameter F (equations 13 and 14 above). In the non-linear case these are, in addition, the parameter [illegible] (reference (2)), applicable to the quasi-static deformation of materials with temperature dependent properties, and the new parameter D (equations 10 and 11 above).

The experiments show how intricate the mechanical behavior of an extremely simple material model with a single viscoelastic relaxation time can become when reasonable allowances are made for the temperature dependence of its properties. In the analog it seems to be possible to make allowances for the geometric changes which accompany many deformations and the normal heterogeneity and anisotropy of many materials.

Among the published mechanical experiments known to the authors, those of VonKarmen and Duwez (7) on the drawing of copper wires seem most relevant to to the above discussion. The present study must be distinguished from others often called dynamic (Reference 8) which apply to isothermal experiments with massless materials. Here inertial and thermal effects are included.

ACKNOWLEDGEMENT

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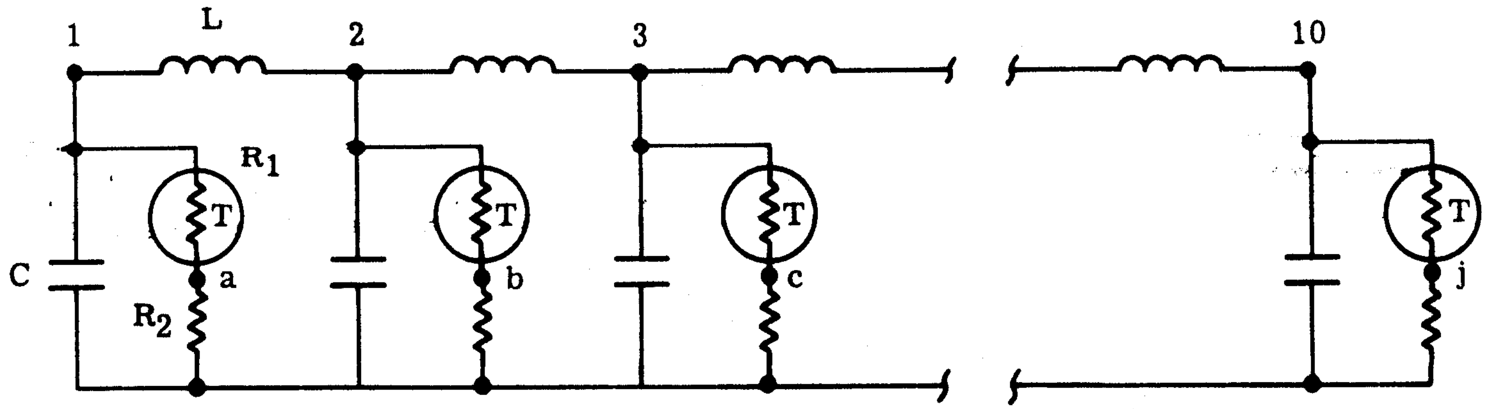

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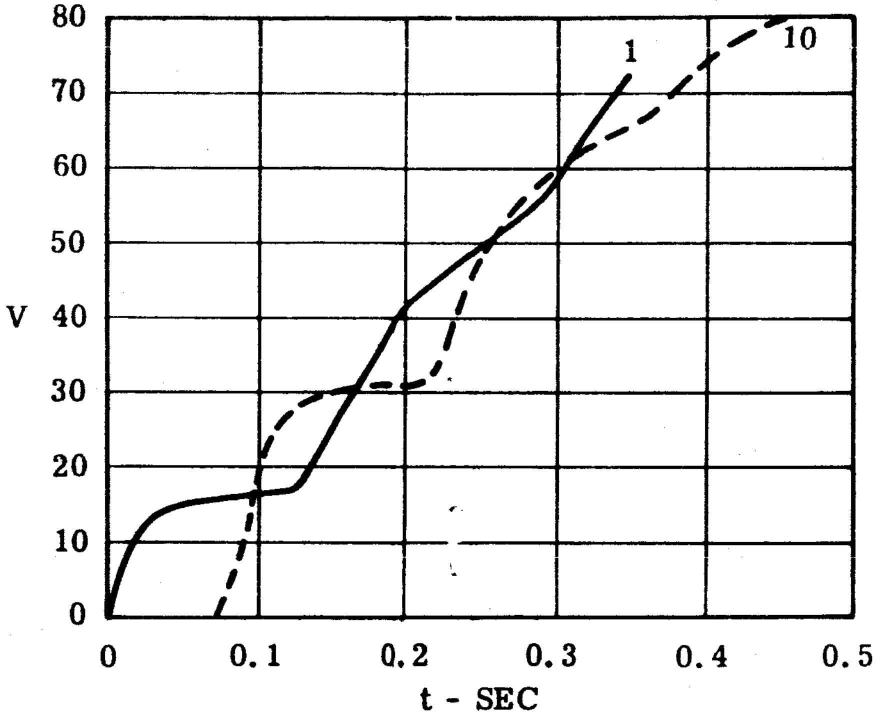

FIGURE 2. VOLTAGE HISTORIES IN ELASTIC ANALOG LINE AT CONSTANT APPLIED CURRENT

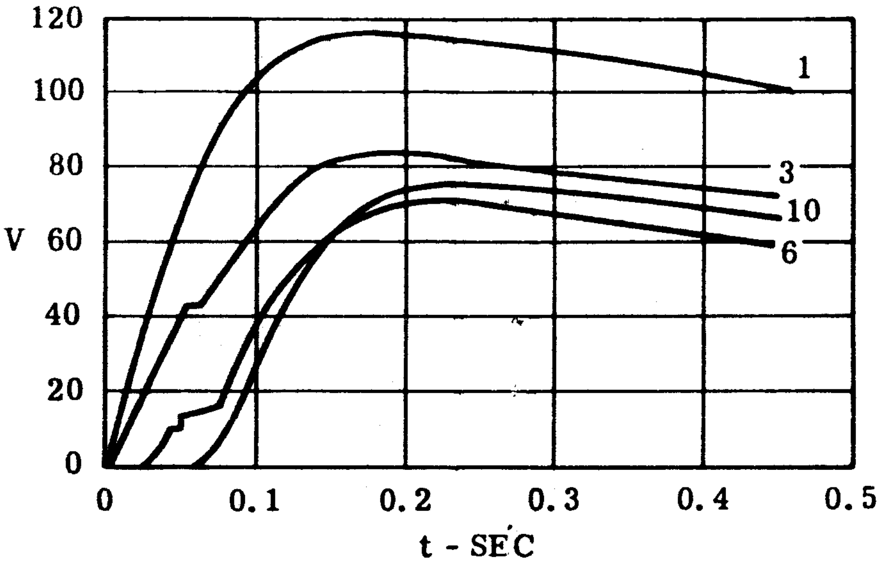

FIGURE 3. VOLTAGE HISTORIES IN VISCOELASTIC ANALOG LINE AT CONSTANT APPLIED CURRENT

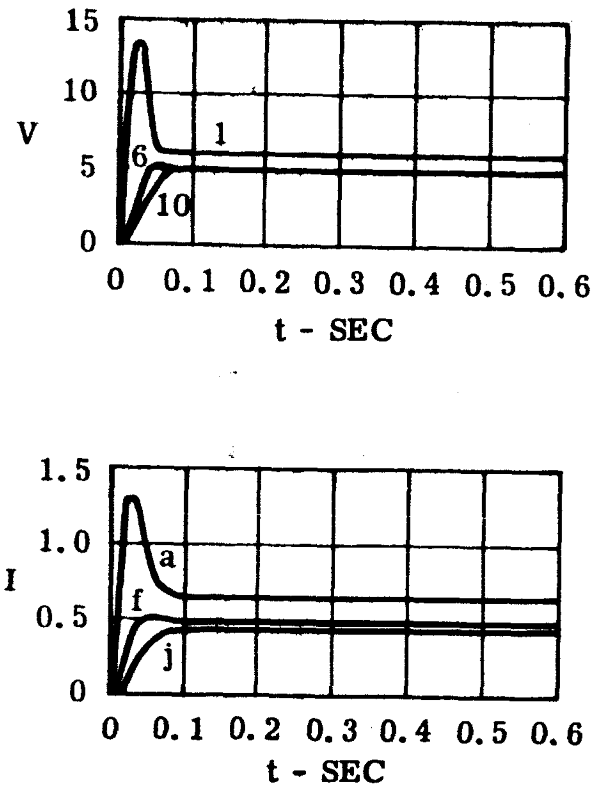

FIGURE 4. VOLTAGE AND CURRENT HISTORIES IN VISCOUS ANALOG LINE AT LOW CONSTANT APPLIED CURRENT

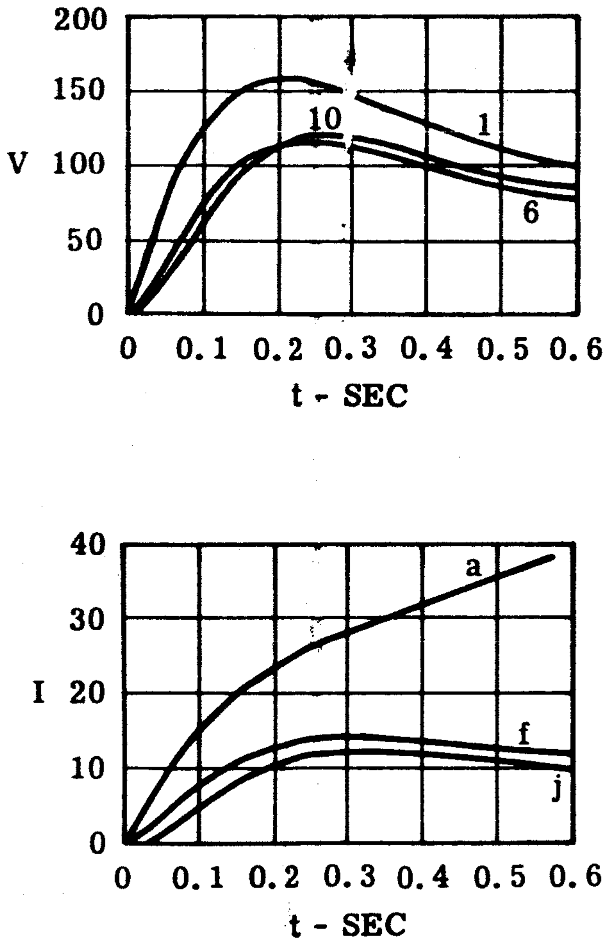

FIGURE 5. VOLTAGE AND CURRENT HISTORIES IN VISCOUS ANALOG LINE AT INTERMEDIATE CONSTANT APPLIED CURRENT

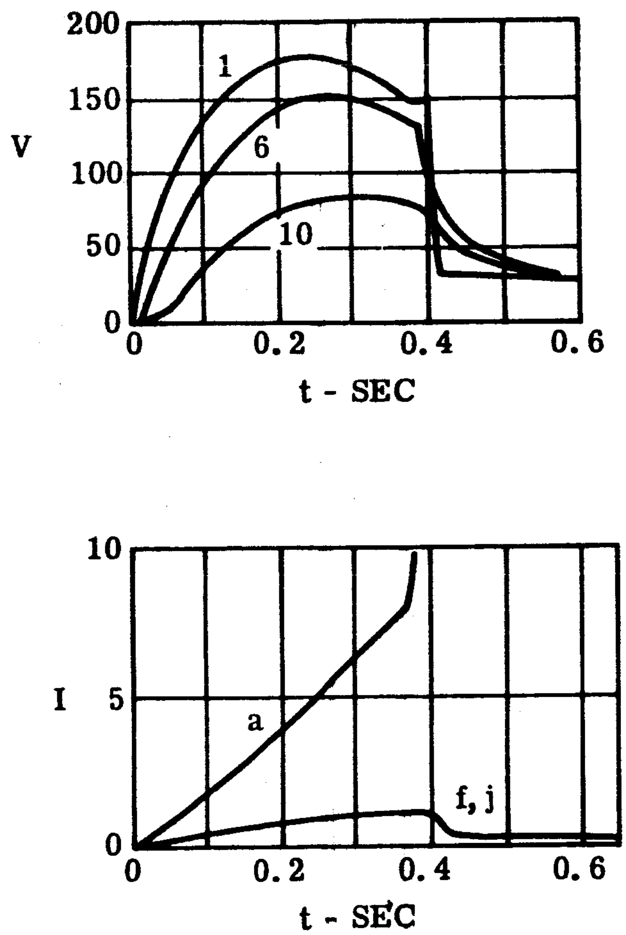

FIGURE 6. VOLTAGE AND CURRENT HISTORIES IN VISCOUS ANALOG LINE AT HIGH CONSTANT APPLIED CURRENT

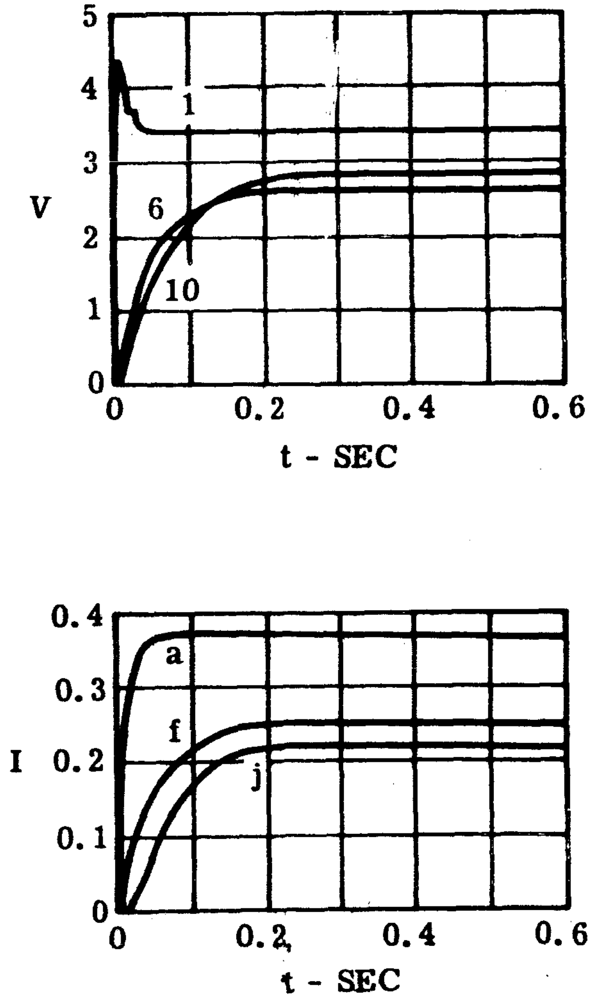

FIGURE 7. VOLTAGE AND CURRENT HISTORIES IN VISCOUS ANALOG LINE AT LOW CONSTANT APPLIED VOLTAGE

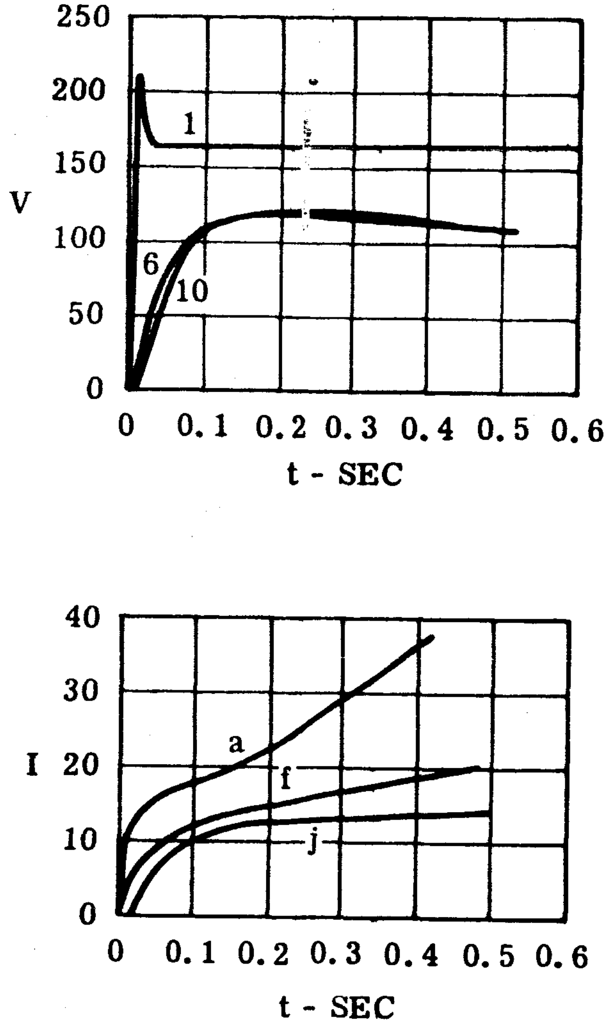

FIGURE 8. VOLTAGE AND CURRENT HISTORIES IN VISCOUS ANALOG LINE AT HIGH CONSTANT APPLIED VOLTAGE

ABSTRACT

As part of a continuing study of the mechanical behavior of materials with temperature dependent properties, dynamic responses of an ideal viscous material are considered. An electric analog circuit containing thermistors and ideal inductors is presented. The characteristics of this circuit simulate the highly non-linear characteristics of the model of the material. The experiments show how the conversion of mechanical work to heat can lead to stress and time dependent apparent viscosities and to thermal instability in velocity driven Couette flow. When the boundary velocity is sinusoidal the heating can lead to the appearance of a pseudo-elasticity as well as to thermal instability. The thermal stability of the system is shown to depend on the frequency as well as the amplitude of the excitation. A group of similarity parameters are described which are ratios of characteristic times for heating, velocity diffusion, thermal diffusion and loading. These provide a description of the model mechanical experiment and make the connection with the electrical analog.

THERMISTOR ANALOG STUDY OF DYNAMIC SHEAR IN AN IDEAL VISCOUS MATERIAL

INTRODUCTION

This report is one of a series describing exploratory studies of the mechanical behavior of materials with temperature dependent properties. With such materials, the heat produced in an experiment can have a strong influence on its outcome. In this work, a model material is considered in which the viscosity is related to the temperature through an energy of activation. The equations describing the behavior of the model are strongly non-linear so that general closed form solutions are not available at this time. The approach taken here involves experiments with electric analog circuits containing resistors (thermistors) in which the temperature dependence of the resistance can be described in terms of an energy of activation.

In earlier studies (1), analog circuits containing thermistors and capacitors were applied to the study of quasi-static deformations of model viscous and viscoelastic materials. Inductors were then added to the circuits (2) to permit the study of the dynamics of the deformations.

While selected commercial thermistors and capacitors are nearly ideal circuit elements, commercial inductors have some resistance and their values depend somewhat on the current. In the present work this problem was overcome by simulating ideal inductors with analog computer elements.

Except for the use of thermistors instead of ordinary resistors, the circuits are similar, in principle, to those described by Mason (3). That is, line inductances and shunt resistances, respectively, simulate the distributed mass and viscosity in the material. To simplify the problem, the model is an infinite slab of an incompressible viscous material in which none of the energy of a shear deformation is stored. Thus, the experiments are preliminary to the study of more complicated systems involving elasticity and in which more than one dimension is considered. The results, however, do provide some insights into the influence of the requirement for energy conservation on the mechanical behavior of materials.

In the experiments, the responses of the model to the application of a shear velocity at one of its boundaries are observed. Both abruptly applied constant velocities and sinusoidally varying velocities are considered. In the former case, as a result of the non-linearity of the material, the observed stress at the stationary boundary becomes a non-linear function of the applied velocity. At higher applied velocities, thermal instability can occur in the material.

In the sinusoidal case, the non-linearity can lead to the appearance of a pseudo-elasticity in the material. When the frequency is high, even at low amplitudes, thermal instability can develop. In real experiments with liquids this instability could be expected to lead to cavitation. At low amplitudes and low frequencies, when the behavior of the material is linear, a pseudo-elasticity also appears in the material if the data are reduced using the quasi-static methods described by Ferry (4).

BACKGROUND

The usefulness of the electric analog in mechanical studies depends on the similarity of the equations describing the voltage and current history in the circuit and those describing the stress and strain rate history in the mechanical model. The availability of nearly ideal circuit elements and the convenience and accuracy of electrical measurements are also considerations.

Briefly, the local instantaneous force balance in an infinite slab of material with viscosity η and density ρ , subjected to a shear in the plane of its boundaries is given by

$$\frac{\partial}{\partial y}\left(\eta \frac{\partial u}{\partial y}\right) = \rho \frac{\partial u}{\partial t} \tag{1}$$

in which u is the local velocity in the x direction, y is the space variable through the thickness of the slab and t is the time. This is identical to the equation for the local line current, i, in the circuit with lumped parameters shown schematically in figure 1,

$$\frac{\partial}{\partial y}\left(R \frac{\partial i}{\partial y}\right) = L \frac{\partial i}{\partial t} \tag{2}$$

in which R is the shunt resistance per unit length (per loop) in the y direction and L is the inductance per unit length. When η and ρ or R and L are constants, equations 1 and 2 reduce to the familiar linear diffusion equation for which solutions are known. On the other hand, when η or R depends on t and y, as will be the case for temperature dependent materials, general solutions in closed form are not available.

In the model material the flow process, on a molecular scale, is assumed to be a biased diffusion so that the dependence of the viscosity on temperature is given by

$$\eta = \eta_0 \exp E_A \left(\frac{1}{T} - \frac{1}{T_0} \right) \tag{3}$$

in which η_0 is the viscosity at an initial or reference temperature, T_0 , and E_A is an activation energy divided by the Boltzmann constant. In the thermistor a similar expression can be written for the resistance R.

$$R = R_0 \exp E_A \left(\frac{1}{T} - \frac{1}{T_0} \right) \tag{4}$$

The instantaneous, local temperature T which determines the instantaneous local viscosity (or resistance) is determined by the local energy balance condition

$$\sigma \frac{\partial u}{\partial y} = c \frac{\partial T}{\partial t} - k \frac{\partial^2 T}{\partial y^2} \tag{5}$$

or

$$V I = c \frac{\partial T}{\partial t} - k \frac{\partial^2 T}{\partial y^2} \tag{6}$$

in which c and k are respectively the temperature independent volumetric heat capacity and thermal conductivity. The local stress, σ , (or voltage, V) is related to the local viscosity and velocity gradient (or resistance and shunt current) by the equations

$$\sigma = \eta \frac{\partial u}{\partial y} \tag{7}$$

$$V = RI \tag{8}$$

A detailed discussion of these equations is given in references (5) and (1). In summary, however, at low levels of stress (or voltage) the heating computed from equations 5 and 6 will be negligible. At higher stress levels the temperature will change significantly and the viscosity or resistance will decrease. The extent of the temperature rise depends on the value of a non-dimensional parameter, δ , which is the ratio of the rate of heating in the slab to the rate of cooling. The heating rate can be defined in terms of a representative time, t_∞ , which is the time that would be required for the viscosity to decay to one half its initial value in a quasi-static, adiabatic experiment (5),

$$t_\infty = \frac{T_0^2 c \ell^2}{E_A \eta_0 u_0^2} = \frac{T_0^2 c \ell^2}{E_A R_0 i_0^2} \tag{9}$$

in which ℓ is the thickness of the slab or the number of loops in the line and u_0 and i_0 are respectively the applied boundary velocity or applied current. The cooling rate can be defined for the case of isothermal boundaries in terms of a representative time, t_c, which is the time that would be required for the temperature to decay to 1/10 of its initial value after the heating is discontinued. Ideally, this would be given by

$$t_c = \frac{c\ell^2}{4k} \tag{10}$$

The key parameter for heating, ζ, can be obtained by comparing these times.

$$\zeta = \frac{t_c}{t_\infty} = \frac{E_A \eta_0 u_0^2}{4T_0^2 k} = \frac{E_A R_0 i_0^2}{4T_0 k} \tag{11}$$

Three more non-dimensional parameters for the dynamic system are useful. One can be generated by taking the ratio of a representative velocity propagation time to the representative heating time. When this number is low the propagation occurs in an essentially isothermal medium. When the number is high the propagation occurs in a medium of changing temperatures. A typical propagation time (from the Fourier number) is given by

$$t_p = \frac{\ell^2 \rho}{\eta_0} = \frac{\ell^2 L}{R_0} \tag{12}$$

A definition of this similarity criterion is therefore

$$P = \frac{t_p}{t_\infty} = \frac{E_A \rho u_0^2}{T_0^2 c} = \frac{E_A L i_0^2}{T_0^2 c} \tag{13}$$

Another parameter is applicable when the boundary of the slab is subjected to an alternating velocity. It is the ratio of the velocity period to the heating time. When this is low, any particular velocity cycle can be considered to be isothermal. When it is high, significant temperature changes can occur during a single cycle. This parameter is

$$C = \frac{1}{f t_\infty} = \frac{E_A \eta_0 u_0^2}{f T_0^2 c \ell^2} = \frac{E_A R_0 i_0^2}{f T_0^2 c \ell^2} \tag{14}$$

in which f is the frequency of the excitation in cycles per second.

The third parameter is also applicable in the alternating case. It is the ratio of the propagation time to the period

$$F = f t_p \tag{15}$$

This parameter provides an index of the phase differences at the two boundaries of the model.

DESCRIPTION OF APPARATUS

The analog circuit used to simulate the slab of viscous material is shown schematically in figure 1. The actual circuit used is shown in figure 2. This involved some development work that will not be discussed here. The principle of the ideal inductor depends on the computation of the time integral of the local applied voltages to obtain the local line currents

$$i_N = \frac{1}{L} \int (V_N - V_{N+1})\, dt$$

The local shunt current is the difference between local line currents

$$I_N = i_N - i_{N+1}$$

and the local voltage is given by

$$V_N = I_N R_N$$

The circuit is shown in a mode that permits regulation of one boundary velocity with the other boundary held stationary. Other types of mechanical boundary conditions can be simulated by making minor changes. The parameters of this circuit can be varied continuously over a wide range so that a convenient time scale can be used.

In order to simulate heat conduction in the slab, disc type thermistors were fitted with thin brass electrodes and stacked using thin plastic film spacers so that they were insulated from one another electrically but were in thermal contact. The cylindrical stack was covered with thermal insulation except on its ends at which the thermal boundary conditions could be controlled. This configuration is thermally analogous to the infinite slab of model material except for the heat conduction through the side insulation and the electrodes.

The thermistors had a nominal resistance of 10,000 ohms at 25°C and temperature coefficients of 5.5% per degree centigrade. Each thermistor was 0.55 cm in diameter and 0.127 cm thick. Its weight was 0.100 grams and its heat capacity was approximately 0.60 joules per gram per degree centigrade. Its thermal conductivity was estimated to be 0.02 joules/sec cm^2 °c/cm.

The connection between the experiment with the analog and that with a mechanical model is made in terms of the parameters mentioned above. The amplifiers were scaled so that the effective value of L was one henry and

R_o was 4 ohms. From this data we have

$$t_\infty = \frac{T_0^2 c \ell^2}{E_4 R_0 i_0^2} = \frac{(295)^2 (.06)(5)^2}{(2250)(4)\ i_0^2} = \frac{15}{i_0^2}\ \text{sec}$$

$$t_c = \frac{c\ell^2}{4k} = \frac{(.06)(5)^2}{(4)(2.1)} = 0.18\ \text{sec}$$

$$\zeta = \frac{t_c}{t_\infty} = 0.0117\,(i_0)^2$$

$$t_p = \frac{\ell^2 L}{R_0} = \frac{(5)^2\, 1}{4} = 6.25\ \text{sec}$$

The most serious approximation in these calculations is that of t_c where heat losses from the sides of the stack and the thermal resistance of the plastic film have been neglected.

EXPERIMENTAL

CONSTANT BOUNDARY VELOCITY

The application of a current to the input terminals of the network is analogous to the application of a velocity to one boundary of the slab. When a constant current, i_0 , is suddenly applied, a high local voltage appears at the input terminals. This is relieved as the current propagates down the line. If the end of the line is open, corresponding to a stationary far boundary on the slab and there is no significant heating, steady uniform voltages and shunt currents develop

$$I_N = \frac{i_0}{l} = \frac{V_0}{R_{0N}} \qquad (16)$$

This is analogous to

$$\frac{\partial u}{\partial y} = \frac{u_0}{l} = \frac{\sigma}{\eta_0} \qquad (17)$$

in the mechanical model. This effect is shown in the plot (figure 3a) of the various values of I_N against time for a scaled value of i_0 = 2.5 amperes ($\mathcal{S}$ = 0.07, P = 0.40).

Results obtained at higher currents are shown in figure 3b. Here i_0 = 100, $\mathcal{S}$ = 117 and P = 600. The non-linearity due to heating appears as a retardation of the propagation process and a time dependent distortion of the velocity profile. Notice that heat conduction finally produces a lower velocity gradient at the boundaries than in the center of the slab. In this case, the voltage at the stationary boundary never reaches $\frac{i_o R_o}{\ell}$ so that even the initial stress measured in a Couette type viscometer would not give a true measure of the viscosity.

At even higher current levels, which are not accessible in this experiment but which were observed in (2), a thermal instability develops in the slab which would perhaps correspond to fracture in a viscous material!

<u>SINUSOIDAL BOUNDARY VELOCITY</u>

Studies of the responses of materials to sinusoidal stress and strain programs are frequently made because they provide information about the time dependence of the properties. Amplitude and phase relationships are interpreted in terms of viscoelastic spectra characteristic of the material (4).

When dynamic effects are taken into account, even when there is no heating and the system is linear, phase shifts and attenuation occur in a viscous material. These depend on the value of the parameter F (equation 15) which depends on the sample dimensions. Typical linear results are shown in figure 4a in which the various local line currents are plotted against time. The peak to peak input current was 5, corresponding to an RMS value of about 1.7. This gives $\mathcal{S}$ = 0.035. The frequency was 1 cycle so that $\mathcal{F}$ = 6.25.

Experiments of the type summarized in figure 4a can be used for measuring the viscosity of liquids (3). If, however, heating occurs, non-linearities will develop which could be interpreted in terms of the existence of a shear elasticity in the material. Typical non-linear results are shown in figure 4b for peak to peak input current of 200. This gives $\mathcal{S}$ = 57.5. When this figure is compared with 4a, differences in both attenuation and phase shift can be seen.

The amount of heat generated near the moving boundary is dependent on the frequency as well as the amplitude. That is, the depth of penetration of the shear wave becomes lower as the frequency increases so that the input power is concentrated in a smaller volume of liquid. At high enough frequencies, even at very low amplitudes, heating can be expected. Then, as the viscosity is reduced, the depth of penetration is further reduced so that a regenerative process will be initiated which can produce thermal instability. This could account for the cavitation effects frequently observed when liquids are exposed to acoustical radiation.

CONCLUSIONS

Materials application and development studies are often obstructed by complicated and poorly understood non-linearities in mechanical behavior. As a result, costly, and frequently unsatisfying, empirical test programs must be conducted. In this work, one known source of non-linear behavior is isolated and explored. It is shown that the requirement for the conservation of energy in the material can lead to the appearance of time and stress dependent effects as well as pseudo-elasticity and instability in rheological experiments. The rationalization of these effects in terms of the non-linearity of the system is sometimes simpler than alternative hypotheses that have been advanced. The studies lead to the development of a group of similarity criteria applicable to mechanical experiments. The studies have, so far, involved only simple systems. However, it seems likely that the methods can be generalized so that practical problems of design and development can be considered.

ACKNOWLEDGEMENTS

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CAPTIONS FOR FIGURES

1. The electrical transmission line analogous to a viscous slab, discretized into five lumped element segments. The thermistors are in thermal contact but electrically insulated.
2. The electrical analog computer simulation of the electrical circuit of Figure 1 utilizing thermistors. The scaled viscosity per segment is T/R_S.
3. The normalized shunt currents in the thermistor (analogous to the velocity gradients at five equispaced stations in the viscous slab) for a sudden step input of current i_o (front wall velocity u_o)

   a) $i_o = 2.5$
   $\zeta = .07$
   $P = .40$

   b) $i_o = 100$
   $= 117$
   $P = 600$

4. The normalized line currents in the inductors (analogous to the velocities at different stations in the slab) for a suddenly applied sinusoidal current i_o (front wall velocity u_o).

   a) $i_o(RMS) = 1.7$
   $\zeta = 0.035$
   $F = 6.25$

   b) $i_o = 200$
   $\zeta = 57.5$
   $F = 6.25$

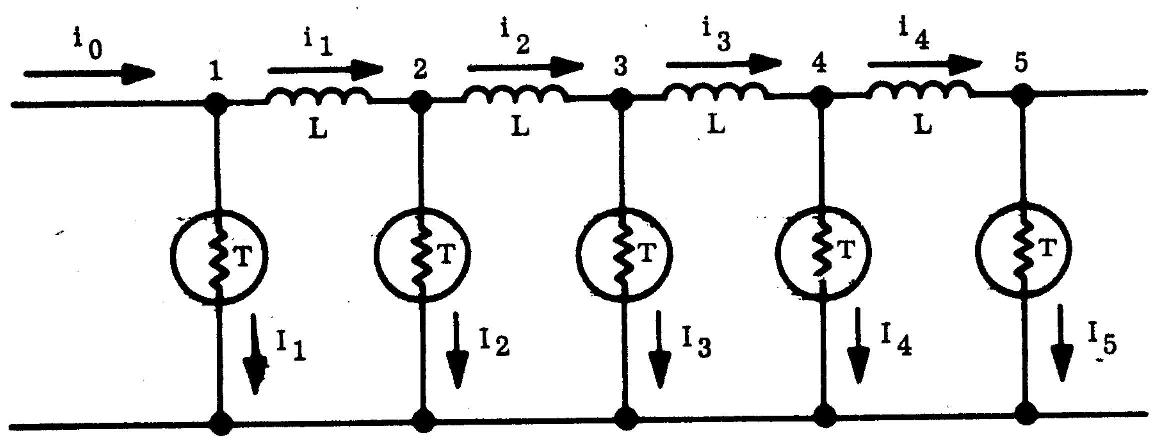
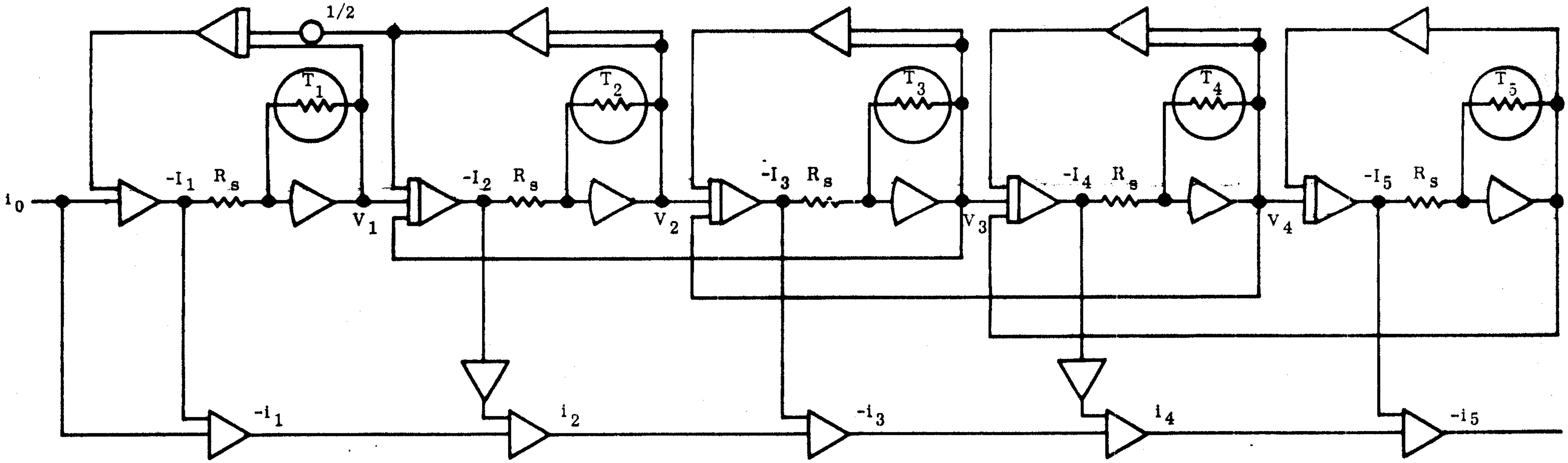


Figure 1



1/2
T_1
T_2
T_3
T_4
T_5
i_0
$-I_1$
R_s
V_1
$-I_2$
R_s
V_2
$-I_3$
R_s
V_3
$-I_4$
R_s
V_4
$-I_5$
R_s
$-i_1$
i_2
$-i_3$
i_4
$-i_5$

Figure 2

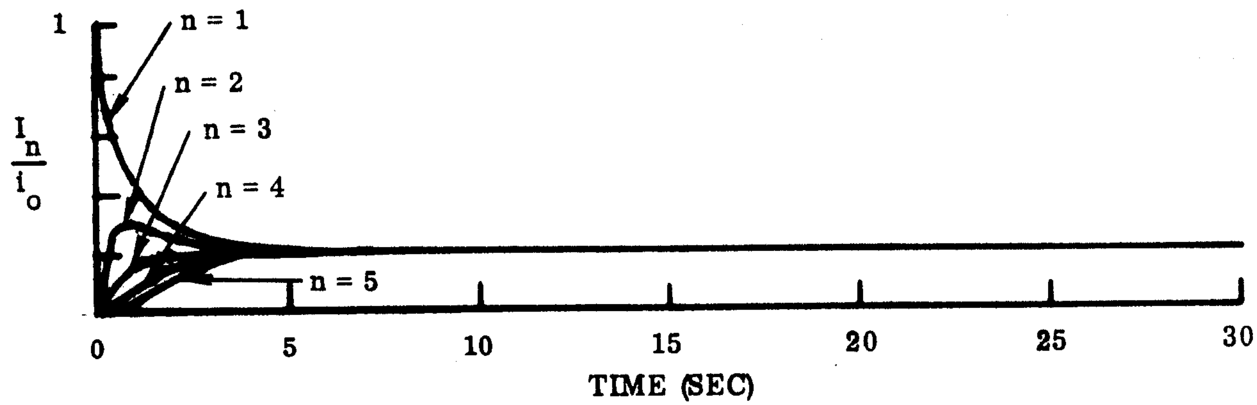


Figure 3a

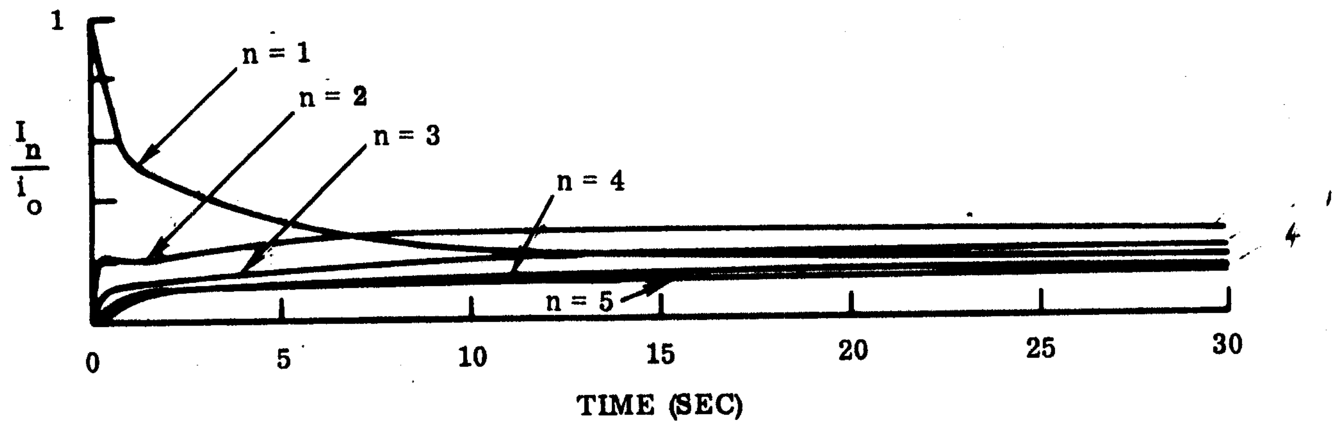
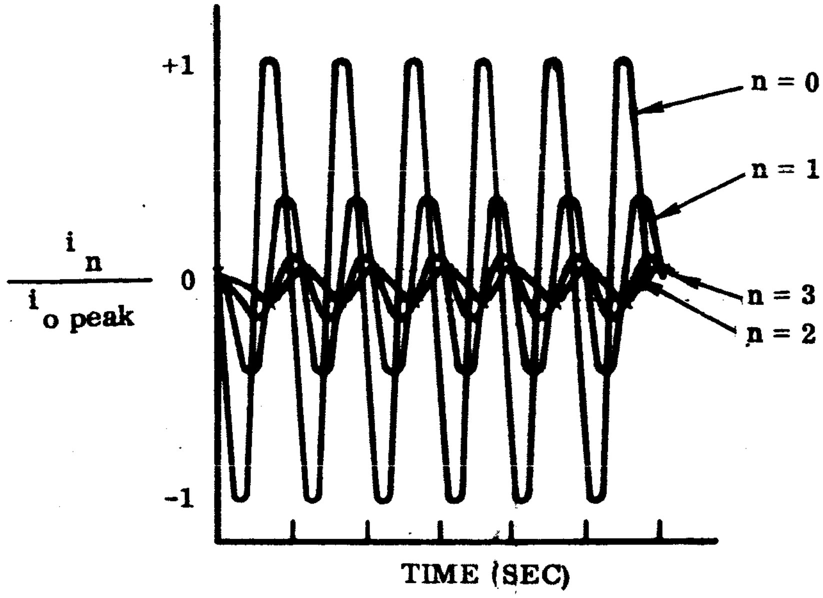


Figure 3b



Figur: 4a

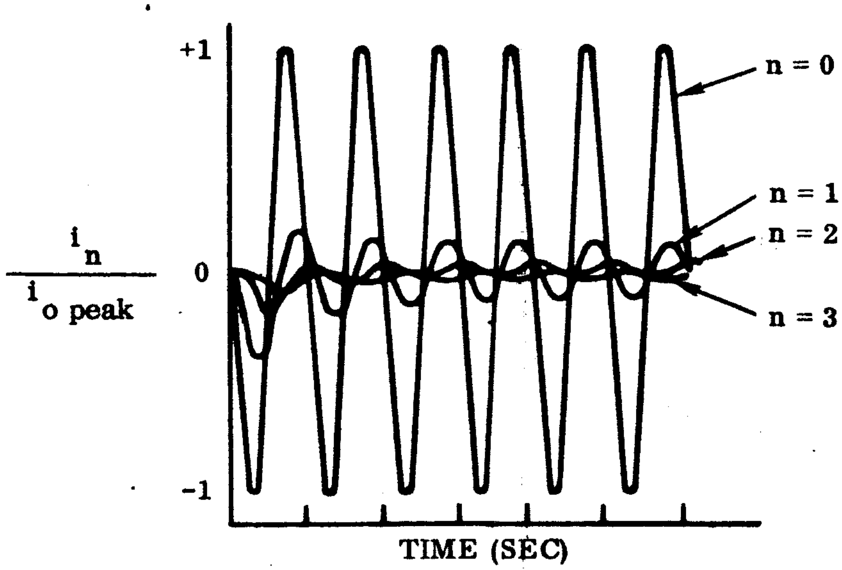


Figure 4b

ABSTRACT

As part of a continuing study of the mechanical behavior of materials which have temperature dependent properties, electric analog experiments on a non-linear Maxwell Model Material are described. The circuit that is used contains thermistor elements which simulate the temperature dependent viscosity of the mechanical model. The inertial and elastic elements are simulated by electronic integrators instead of inductors and capacitors which assures the ideality of the elements and increases the versatility of the analog. The work leads to the definition of a natural "property" of the material which may have some advantages over the use of the viscosity coefficients and elastic moduli alone. The results show how, because of heating, the responses of the model change from elastic to viscous as the severity or duration of the excitation increases. This also leads to stress reduction in the trailing part of the disturbance. Since elements of the analog may be identified with elements of atomic scale models of material, the work relates to material development programs as well as to mechanical design studies.

Analog Study Of the Dynamics of A Non-Linear Maxwell Model Material

Introduction

This report is one of a series describing exploratory studies of the mechanical behavior of materials which have temperature dependent properties. It is an account of experimental studies of one-dimensional, dynamic shear in a non-linear Maxwell Model viscoelastic material. An electric analog method is used which is an extension of that used recently for the study of viscous materials (1). The analog circuit can be made applicable to the general deformation problem. In addition, some of its elements are identifiable with elements of atomic scale models of materials. The work, therefore, relates to material development programs as well as to design studies.

One of the features of the analogs is the use of thermistor elements to simulate the characteristic, temperature-dependent viscosity of certain real materials. Another is the use of precision electronic integrating elements instead of inductors and capacitors to simulate inertial and elastic effects in the model. This assures the ideality of the elements and eliminates the problem of obtaining matched components over a wide range of ratings. Furthermore, the ratings of the components can conveniently be made responsive to the local values of the voltage, current or other relevant parameters of the system.

In the experiments discussed below, a simple model viscoelastic material is considered in which the temperature dependence is concentrated in the viscous elements. This simplification is suggested by the fact that in important classes of real materials the irreversible part of the deformation, whether it be viscous flow or dislocation movement, is a biased diffusion having a characteristic energy of activation and a corresponding exponential dependence of rate on temperature. The behavior of the elastic element depends on the effective shape of the potential valley in which a representative atom is situated or the configurational entropy of a molecular chain either of which is less sensitive to temperature than the diffusion rate.

The quasi-static behavior of a temperature dependent viscous element alone was considered in detail in earlier work in which numerical solutions of the energy equation were presented (2). Those studies established a criterion for apparent departures from Newtonian behavior and showed how time and size effects could arise in viscous materials. The thermal effects could also influence the transition to turbulent flow and produce cavitation. The numerical methods were then applied to the quasi-static deformations of viscoelastic models (3). Later it was shown that equivalent solutions of the quasi-static viscous and viscoelastic problems could be obtained more easily by the use of electric analog circuits containing temperature dependent resistors (4). Phenomena resembling yield and plastic flow arose in a natural way in these models and stick-slip effects were produced.

The use of a transmission line containing thermistors for the study of dynamic viscoelasticity was then described (5). In that study similarity criteria for identifying a particular circuit with a particular mechanical model were defined. In addition, it was shown how the conventional properties of the viscoelastic model could be very sensitive to the stress and strain rate. The transmission lines used there contained commercial inductors which inevitably have some resistance and which are non-linear. Furthermore the commercial components are available over a limited range of ratings and similarly rated elements do not necessarily match very closely. These difficulties were eliminated in another transmission line study of dynamic viscous behavior in which the precision analog integrators were used to simulate inductors (1). That work showed how pseudo-elasticity could arise when the viscous elements became non-linear. In addition a regenerative process was described which could lead to effects resembling acoustic cavitation.

The behavior of the one dimensional Maxwell Model discussed here is one of the simplest available examples of dynamic viscoelasticity. It does not exploit the capability of the method for treating compressible materials with stress, strain or strain rate dependent elasticity or for treating three dimensional models. It is, however, a step in the development of the techniques and it shows how the essential non-linearity of the viscous element can cause a material which exhibits elastic responses at low stresses to behave like a fluid at high stresses. It also leads to an approach to the general characterization of materials which may have some advantages over description in terms of viscosity coefficients and elastic moduli alone.

Discussion Of The Linear Maxwell Model

The dynamic behavior of the linear Maxwell Model has been discussed by Lee and Kanter (6). Their analysis serves as a base line for the consideration of the non-linear model. A slight extension of their discussion is given below. The linear partial differential equation with constant coefficients which is solved in (6) is

$$\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{E}\frac{\partial^2 u}{\partial t^2} + \frac{\rho}{\eta}\frac{\partial u}{\partial t} \tag{1}$$

in which u is the local stress, particle velocity, strain or displacement, x is the Lagrangian space coordinate, t is time and ρ, E and η are respectively the density, elastic shear modulus and viscosity of the material.

Precisely the same equation can be written for the transmission line shown in figure 1

$$\frac{\partial^2 u}{\partial x^2} = LC\frac{\partial^2 u}{\partial t^2} + LG\frac{\partial u}{\partial t} \tag{2}$$

when u is the local current or voltage, L, C, and G are the constant values of the inductance, capacitance and conductance per loop and x is an index of the position of a particular loop in the line. The use of discrete instead of distributed elements introduces some degree of approximation into the applicability of equation 2 to the analog line which is discussed later in the report.

We notice now that the substitution of non-dimensional times and lengths into equation 1 (or equation 2) leads to a single parameter, normalized equation. When

$$\tau = \frac{G}{C} t \text{ or } \frac{E}{\eta} t \quad \text{and} \quad \xi = \frac{x}{\ell} \tag{3}$$

in which ℓ is the length of the line, we have

$$\Gamma^2 \frac{\partial^2 u}{\partial \xi^2} = \frac{\partial^2 u}{\partial \tau^2} + \frac{\partial u}{\partial \tau} \tag{4}$$

in which

$$\Gamma^2 = \frac{C}{G^2 L \ell^2} \text{ or } \frac{\eta^2}{\rho E \ell^2} \tag{5}$$

Thus, the non-dimensional group, or similarity criterion, Γ , emerges as a natural property of the simple, linear Maxwell Model. The use of this number provides a method for identifying a particular transmission line with a particular mechanical model and could be useful for characterizing the mechanical model itself.

Some idea of the significance of this number is given by a consideration of its relationship to more familiar non-dimensional groups that arise in degenerate forms of equation 1. For example if E is very large equation 1 has the form of the diffusion equation

$$\frac{\partial^2 u}{\partial y^2} = \frac{\rho}{\eta} \frac{\partial u}{\partial t} \tag{6}$$

The normalized form of this equation contains the Fourier number, N_F .

$$N_F = \frac{\eta t_0}{\rho \ell^2} \tag{7}$$

On the other hand if η is very large, equation 1 has the form of the elastic wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{E} \frac{\partial^2 u}{\partial t^2} \tag{8}$$

The normalized form of this equation contains another recognizable group involving the velocity of propagation

$$N_E = \frac{E}{\rho} \frac{t_0^2}{\ell^2} \tag{9}$$

The non-dimensional group Γ is seen to be the ratio of N_F^2 to N_E. Notice that the ratio of N_E to N_F gives another number familiar in the study visco-elasticity, Et_0/η .

The physical property associated with Γ is that which may be observed when one boundary of the slab of material is moved abruptly. When Γ is high, oscillations will be produced as the deformation distributes itself through the material. When Γ is low, no oscillations will occur.

Discussion Of The Non-Linear Maxwell Model

The applicability of the analysis given by Lee and Kanter (6) depends on the values of ρ, E and η remaining constant during the experiment. This is never exactly true since any viscous process produces heat. However, if the duration of the experiment is low enough or the heat is lost to the environment fast enough, the change in temperature can be negligible.

The development of a criterion for the temperature rise to be expected in a quasi-static viscous element was described in (2). The criterion appears as a ratio, $\mathcal{G}$, of a representative cooling time, $t_c = \frac{c\ell^2}{k}$, to a representative heating time, $t_\infty = \frac{c\ell^2}{a\eta_0 u_0^2}$

$$\mathcal{G} = \frac{t_c}{t_\infty} = \frac{a\eta_0 u_0^2}{k} \tag{10}$$

in which "a" is the temperature coefficient of the viscosity and is related to the energy of activation for the flow process, η_0 is the viscosity at the initial or reference temperature, u_0 is the relative rate of movement of the boundaries, k and c are, respectively, the temperature independent thermal conductivity and volumetric specific heat of the material.

When $\mathcal{G}$ is low, no heating occurs and the linear approximation is applicable. When $\mathcal{G}$ is high two cases must be distinguished. In one of these the heating time is long in comparison with the mechanical propagation time, so that stress equilibrium is established in an essentially isothermal medium which subsequently increases in temperature. In the other, the heating time is short in comparison with the mechanical propagation time so that the propagation process occurs in a medium of changing temperature. Criteria for making this distinction are discussed in (1).

Discussion of Experiments

The experiments described below were performed using the simulated transmission line shown schematically in figure 2. The symbols of that figure are those used in textbooks on analog computation. The integrators take the place of inductors by generating a "current" proportional to the time integral of a voltage. They take the place of capacitors by generating a voltage proportional to the time integral of a current. The proportionality factors can be variables to simulate compressible, non-Hookean models although this feature was not used in the present experiments.

In an experiment, the effective values of L, C and G were set and a simulated steady current was abruptly applied to one end of the line while the other was kept "open". This is analogous to the abrupt application of a steady velocity to one boundary of the slab of model material while the other is held stationary. Local time dependent line currents and voltages were recorded which correspond to the local velocity and stress in the material. The recording was continued after the input currents were abruptly terminated. Two levels of current were used at each setting. One low enough to show linear behavior and one high enough to show some of the effects of heating in the thermistors. A number of different line settings were used to give representative values of the parameter Γ. The degenerate, ideal viscous, case was discussed in (1). The degenerate, ideal elastic, case was discussed in (5) but is reviewed here.

Figure 3a shows the time dependent currents in a line in which the effective values of L, C and G are respectively 1 henry, 0.03 farads and 0.23 mhos. The value of Γ for the five loop line is 0.15. The input current which is analogous to a boundary velocity was $i_0 = 5$. With the arrangement that was used it is not possible to obtain a good estimate of δ but at this level the behavior of the line was quite accurately linear. At this value of Γ the model is one in which the viscosity is most conspicuous. Attention is drawn to the rapid achievement of the steady condition, the fact that the steady currents are linear functions of the position in the line and the symmetry of the effects at the initiation and termination of the current flow.

Figure 3b is a similar plot for a higher value of the input current ($i_0' = 50$). The linearity of the relationship between the current and line position is destroyed because the thermistor in the first loop in heated more than the others. The symmetry of the initiation and termination effects is also lost because at the time of termination the material is heated and has different properties. Figures 4a and 4b show the local voltage histories in the same two experiments as $5VG/i_0'$. Given the value of a = 0.046 per degree centigrade for the thermistor at the initial temperature and the relationship

$$R = R_0 \exp \frac{a}{T_0 T}(T_0 - T) \qquad (11)$$

these permit the estimation of the local temperature. In the low current case (figure 4a) the lines become horizontal at the value 1.

In the higher current case the voltages drift downward from 1 as a result of heating. Notice that the step in one of the traces on 4a which might be regarded as due to the elasticity in the line is less conspicuous in 4b because the heating reduces the effective value of Γ. The value of δ in the high current runs is not very high even though they were made near the limiting power capability of the circuit configuration. However, the trend of the behavior as the severity of the excitation increases is indicated in these experiments.

The results of experiments with line settings L = 1, C = 0.3, G = 0.2 (Γ = 0.55) are shown in figures 5 and 6. The observed local currents for the low current runs are shown in figure 5a. This line has significant energy storage capacity and the effects of elasticity are clearly shown by the repeated reflections which occur at the boundaries and the associated oscillation of the currents. The linear distribution of steady currents and the symmetry of the initiation and termination effects may be noted here. In the runs at higher current (i_0' = 50) the linearity and symmetry are both lost. Comparison of the voltage traces figures 6a and 6b shows the temperature rise in the high current experiment and the marked attenuation of reflections at the termination of the current flow due to the change in properties accompanying the simulated deformation.

Figures 7 and 8 show the results of experiments with line settings $L = 1$, $C = 30$, $G = 0.2$ ($\Gamma = 5.5$). Here the high energy storage capacity of the line leads to sustained oscillations. Under these conditions more of the current is diverted from the thermistors by the capacitors which reduces the heating in the model. The low current case ($i_o = 8$) show less damping than the high current case ($i_c = 80$). The effect of heating on the symmetry of the initiation and termination effects are also shown.

Figure 9 was generated by a line in which $G = 0$. It therefore simulates a perfectly elastic material. It is presented for two reasons. It indicates how the discretizing of line elements affects its behavior. If the elements were distributed uniformly the voltage would progress upward in steps with vertical "risers" and horizontal "treads". The rounding of the steps is due to the filter action of the line that was used. The other object of presenting this data is to show how, in an elastic material, the stress would rise in stepwise fashion without bound. This emphasizes the importance of viscosity in determining the ultimate stress bearing capability of the material.

Figure 10 gives examples of cross plots of typical data. It shows the wave form at various times including some of the effect of reflection. This is the form in which the analytical results of reference 6 were presented. In the high current case, 10b, a low stress "tail" develops in the wave because of heating.

Plastic Waves in Wires

The results presented above are strictly applicable only in the one dimensional case; that is, to the shear of an infinite slab of the model material in the plane of its boundaries. However, if, in considering the axial deformation of a slender rod, one assumes that stress equilibrium is achieved in each element of length during the propagation process, the line described above is an appropriate analog. This is the problem attacked and assumption made, for example, by von Karman and Duwez (7). The present treatment, however, allows the tangent modulus to be time dependent. In this application, the use of the analog draws attention to the important fact that the irreversible part of the deformation is largely a shear and the reversible part is largely a dilatation.

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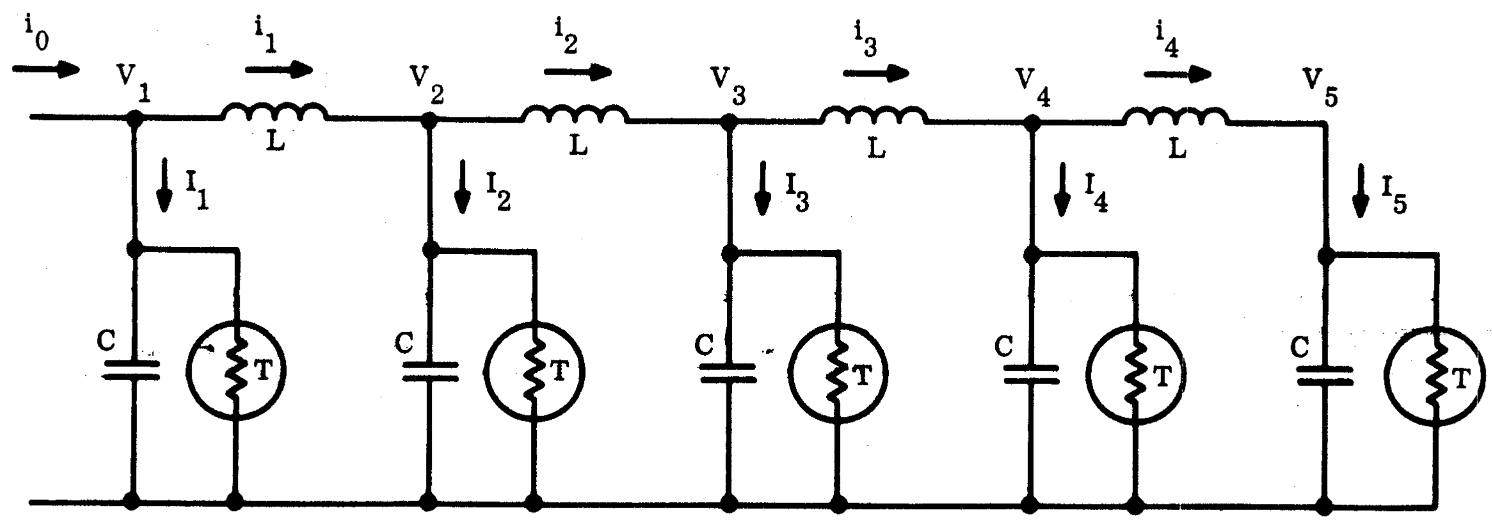


FIGURE 1

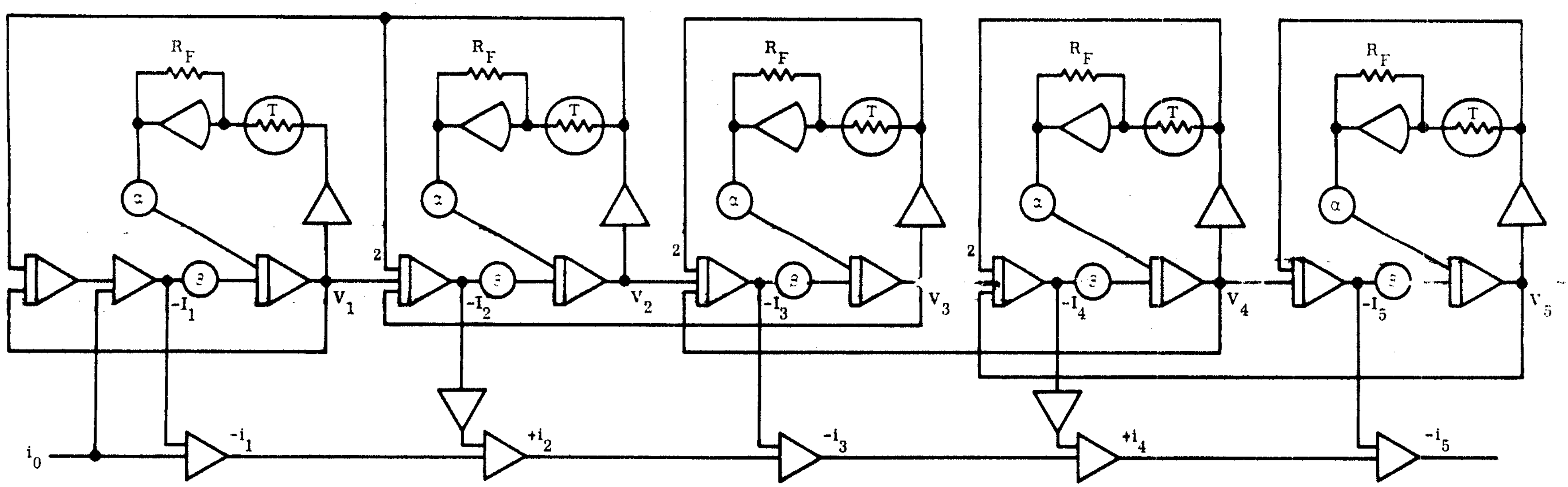


FIGURE 2

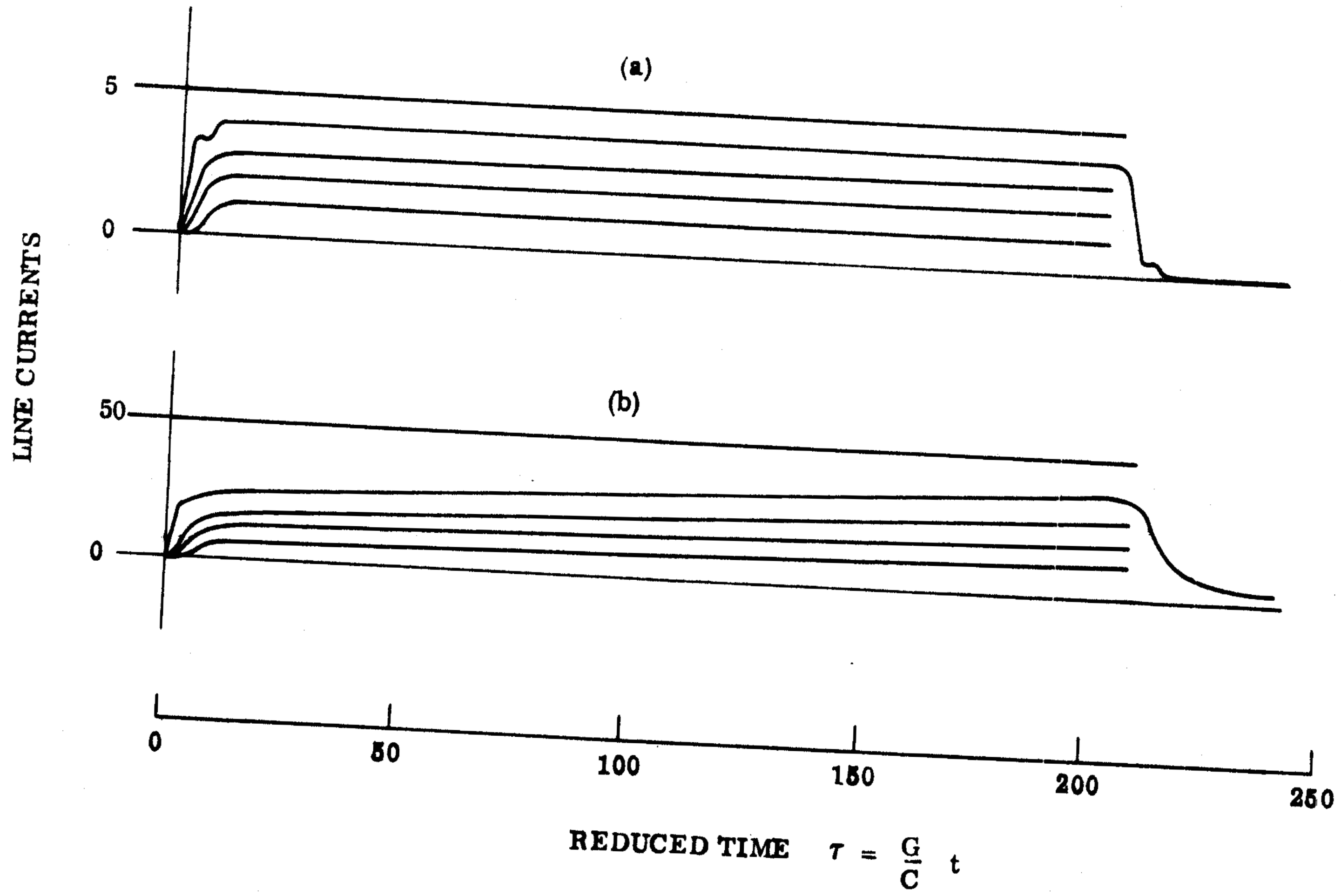

(a)
(b)
5
0
50
0
LINE CURRENTS
0
50
100
150
200
250
REDUCED TIME $\tau = \frac{G}{C}\, t$

FIGURE 3

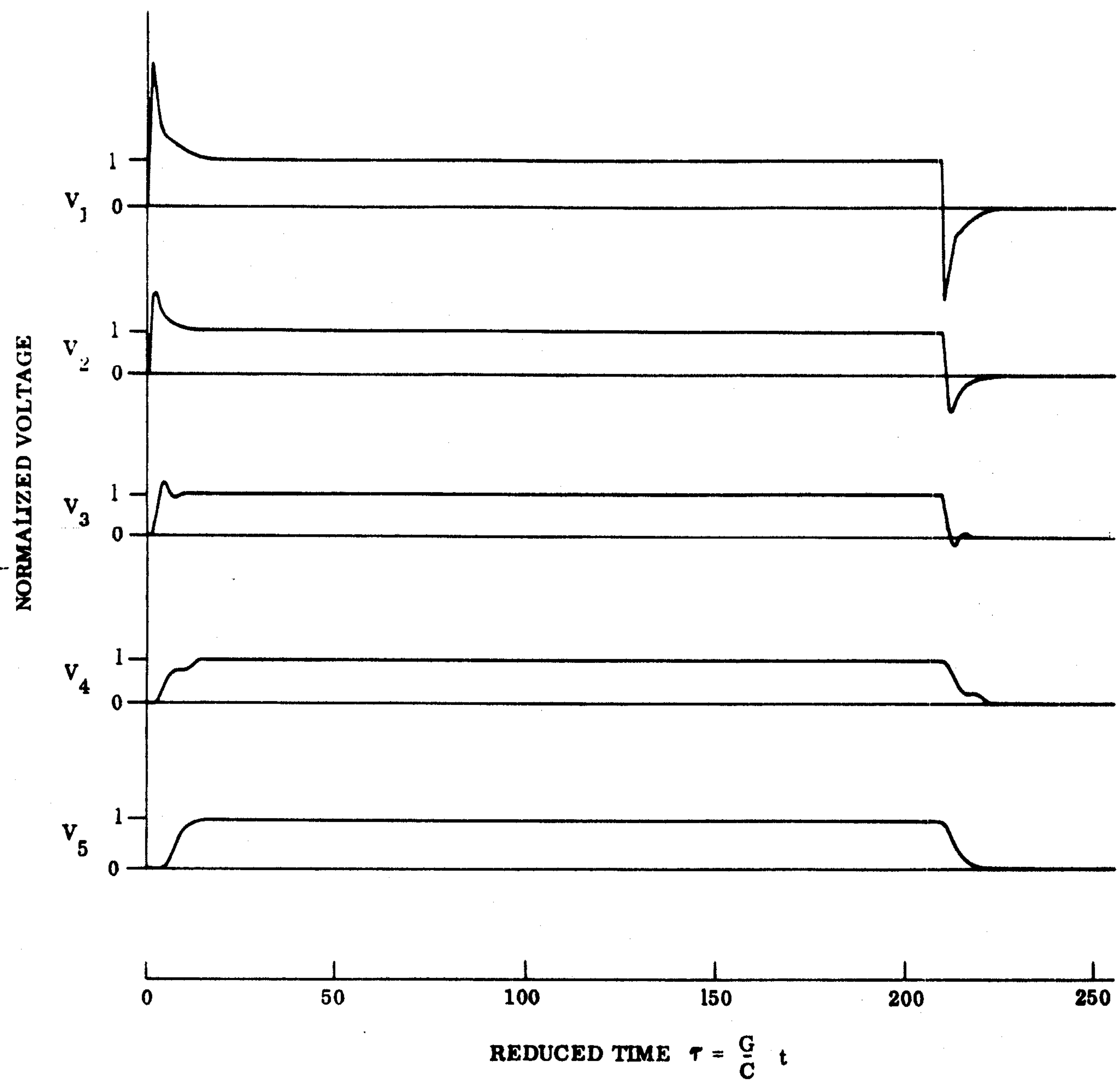

NORMALIZED VOLTAGE
V_1
V_2
V_3
V_4
V_5
1
0
0
50
100
150
200
250
REDUCED TIME $\tau = \frac{G}{C} t$

FIGURE 4 A

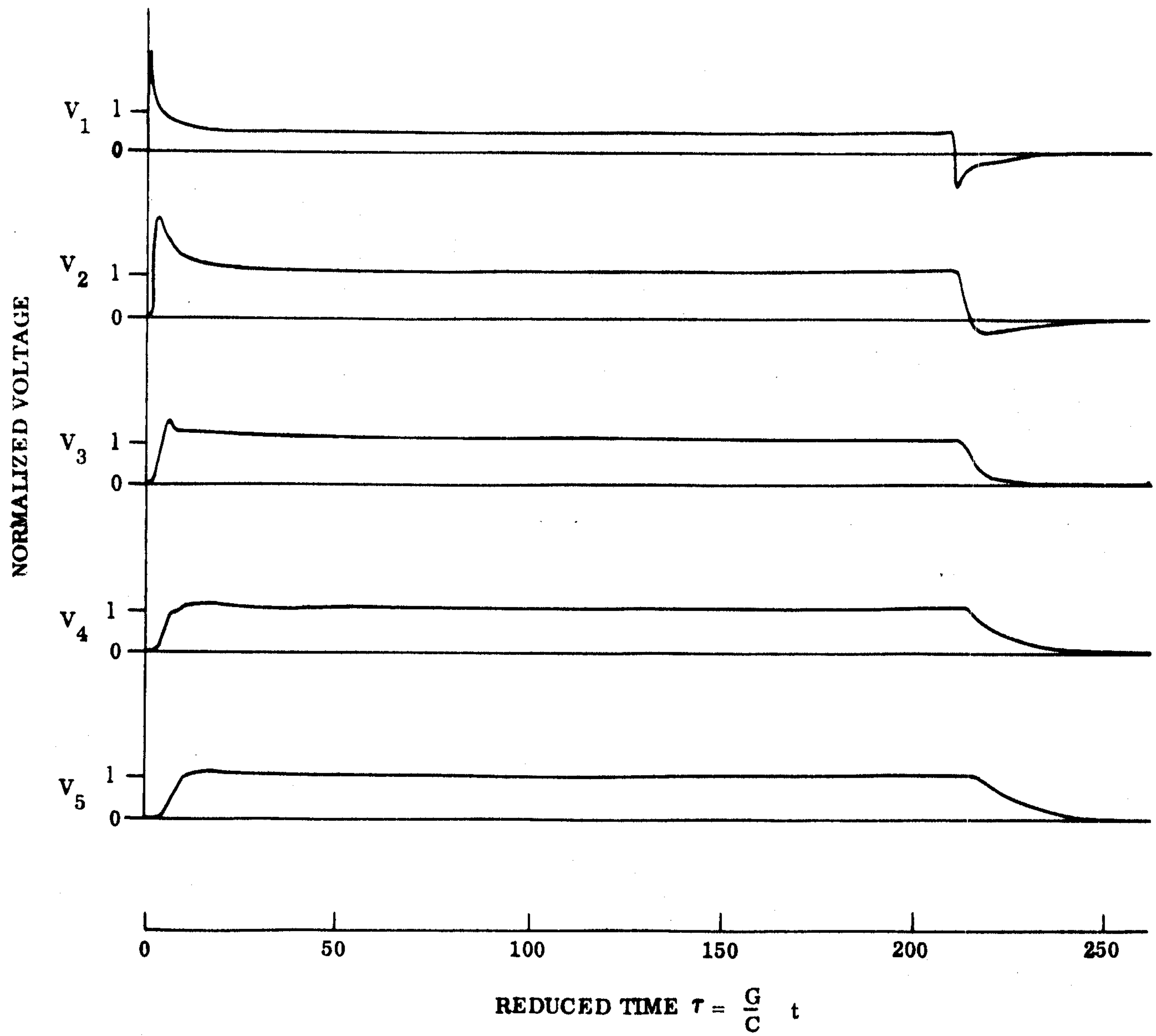

NORMALIZED VOLTAGE
V_1
V_2
V_3
V_4
V_5
1
0
0
50
100
150
200
250
REDUCED TIME $\tau = \frac{G}{C} t$

FIGURE 4 B

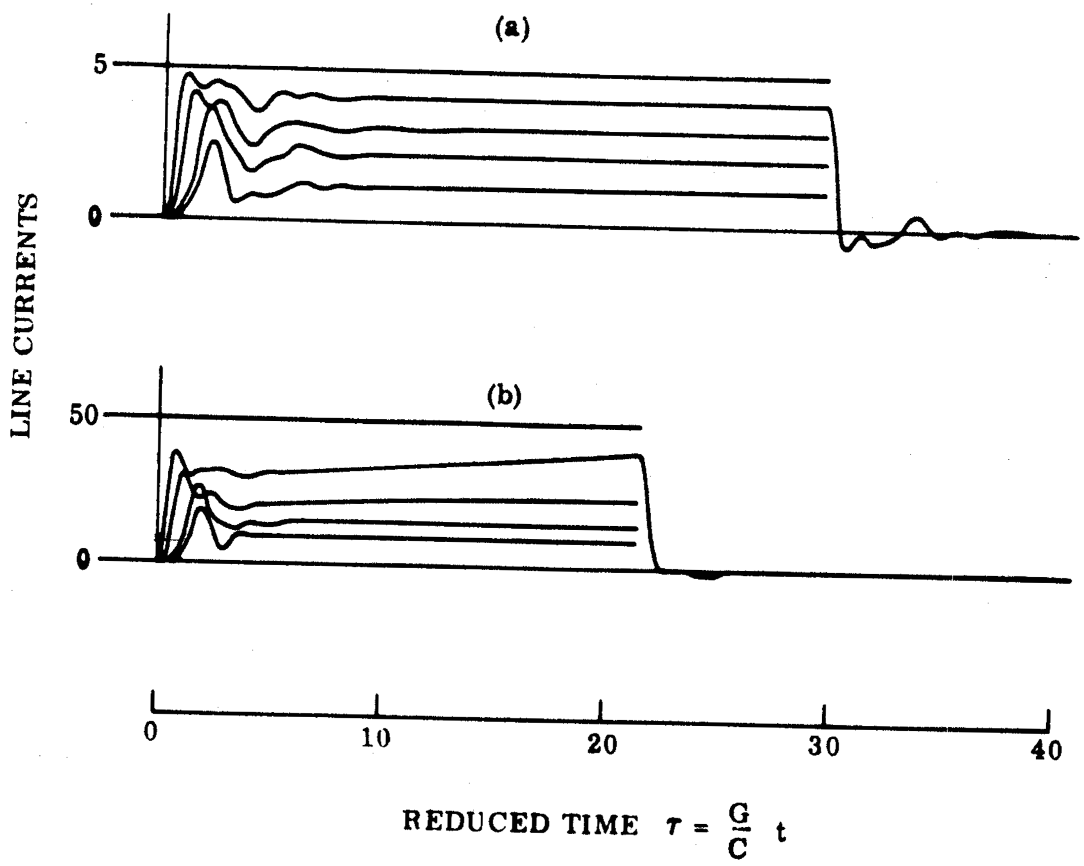


FIGURE 5

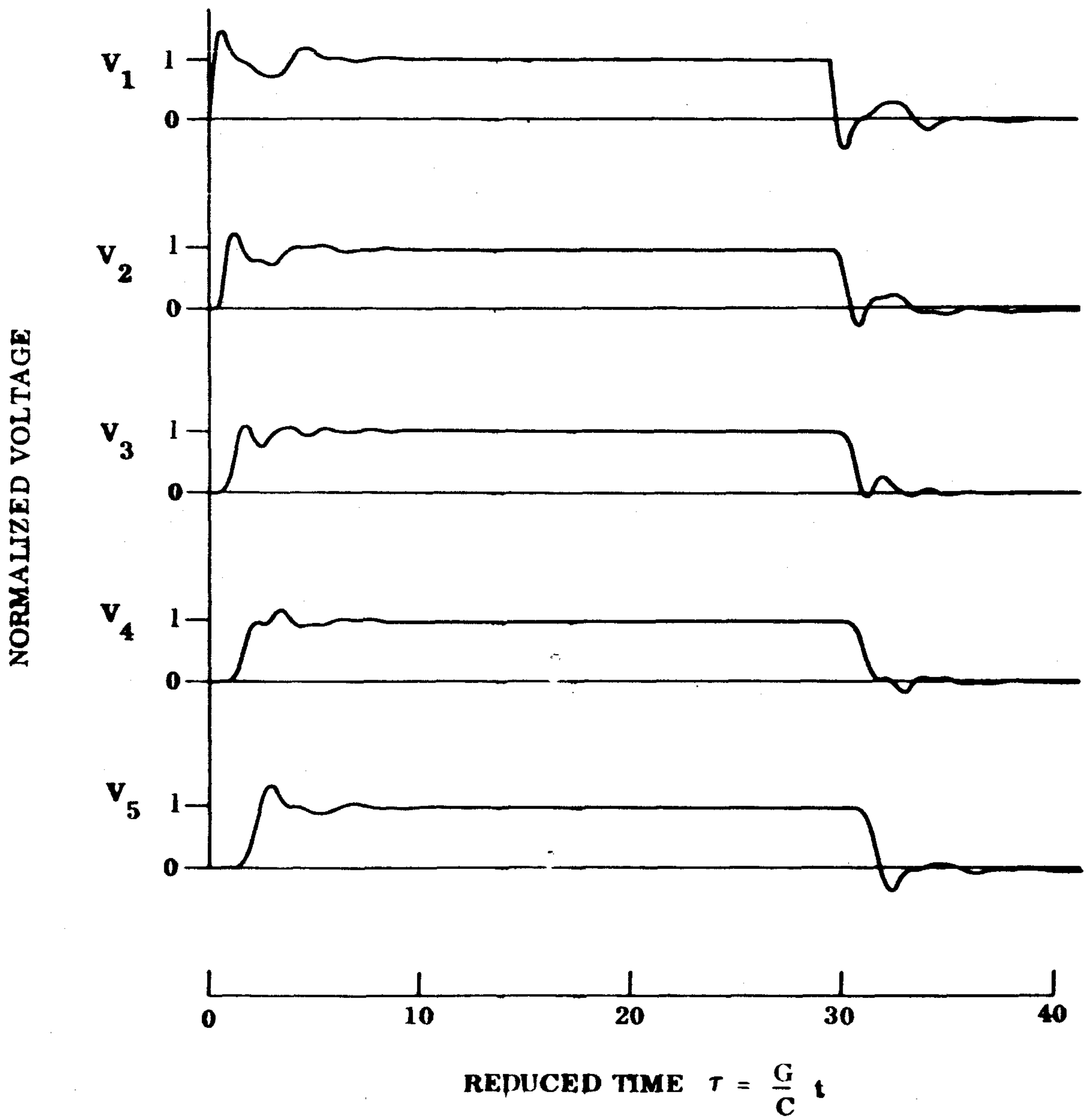
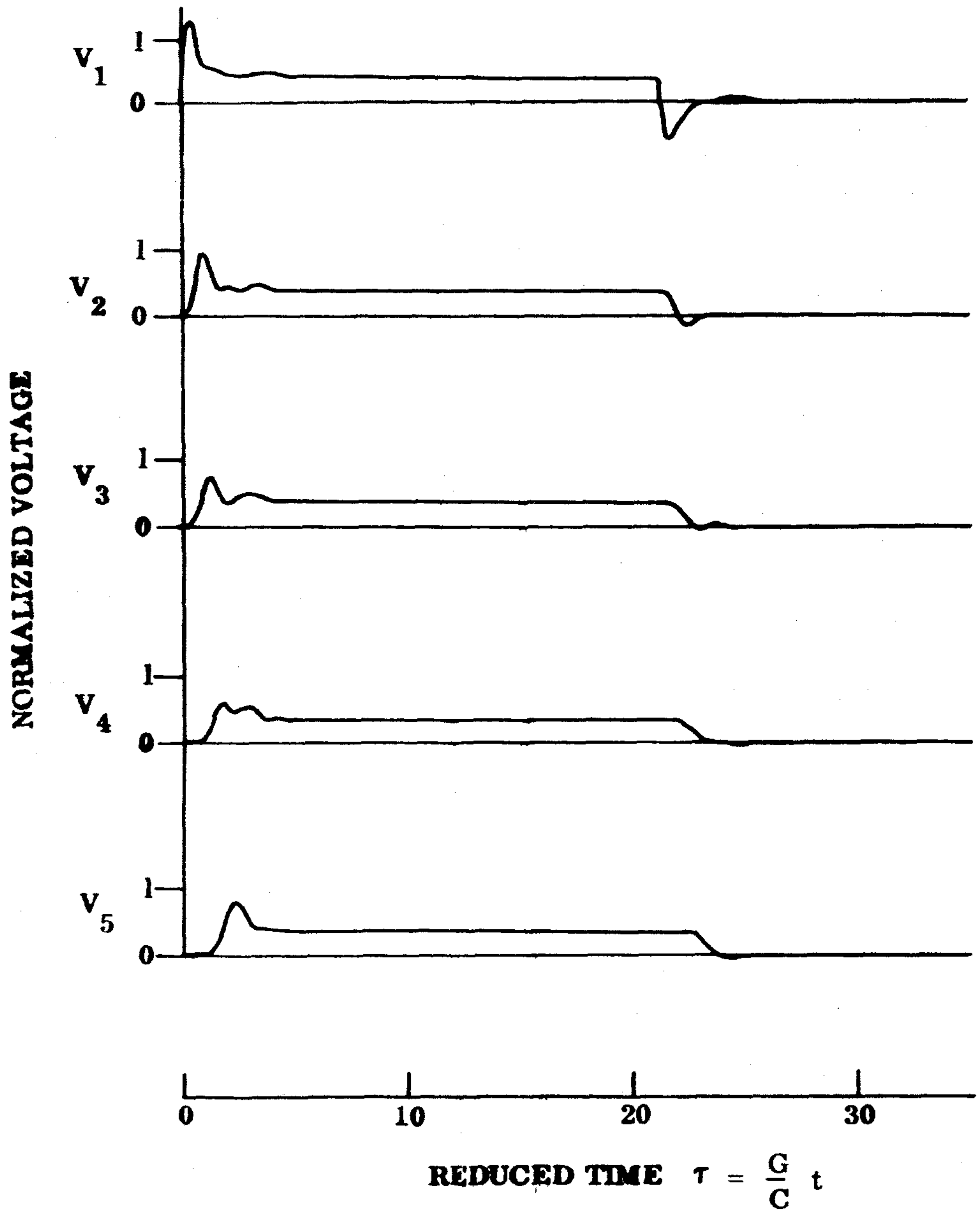

V1
V2
V3
V4
V5
1
0
NORMALIZED VOLTAGE
0
10
20
30
40
REDUCED TIME τ = G/C t

FIGURE 6 A



NORMALIZED VOLTAGE
V1
V2
V3
V4
V5
1
0
0
10
20
30
REDUCED TIME τ = G/C t

FIGURE 6 B

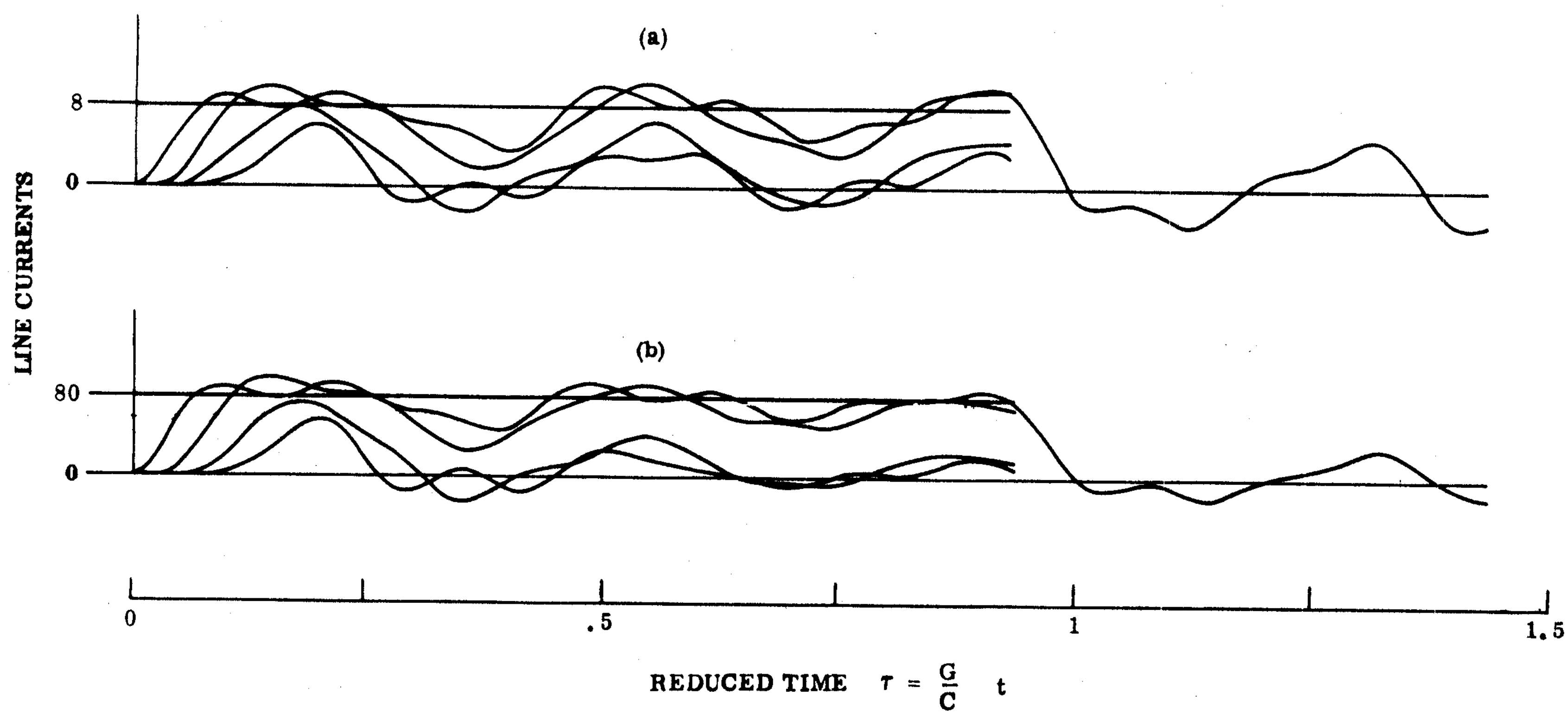


FIGURE 7

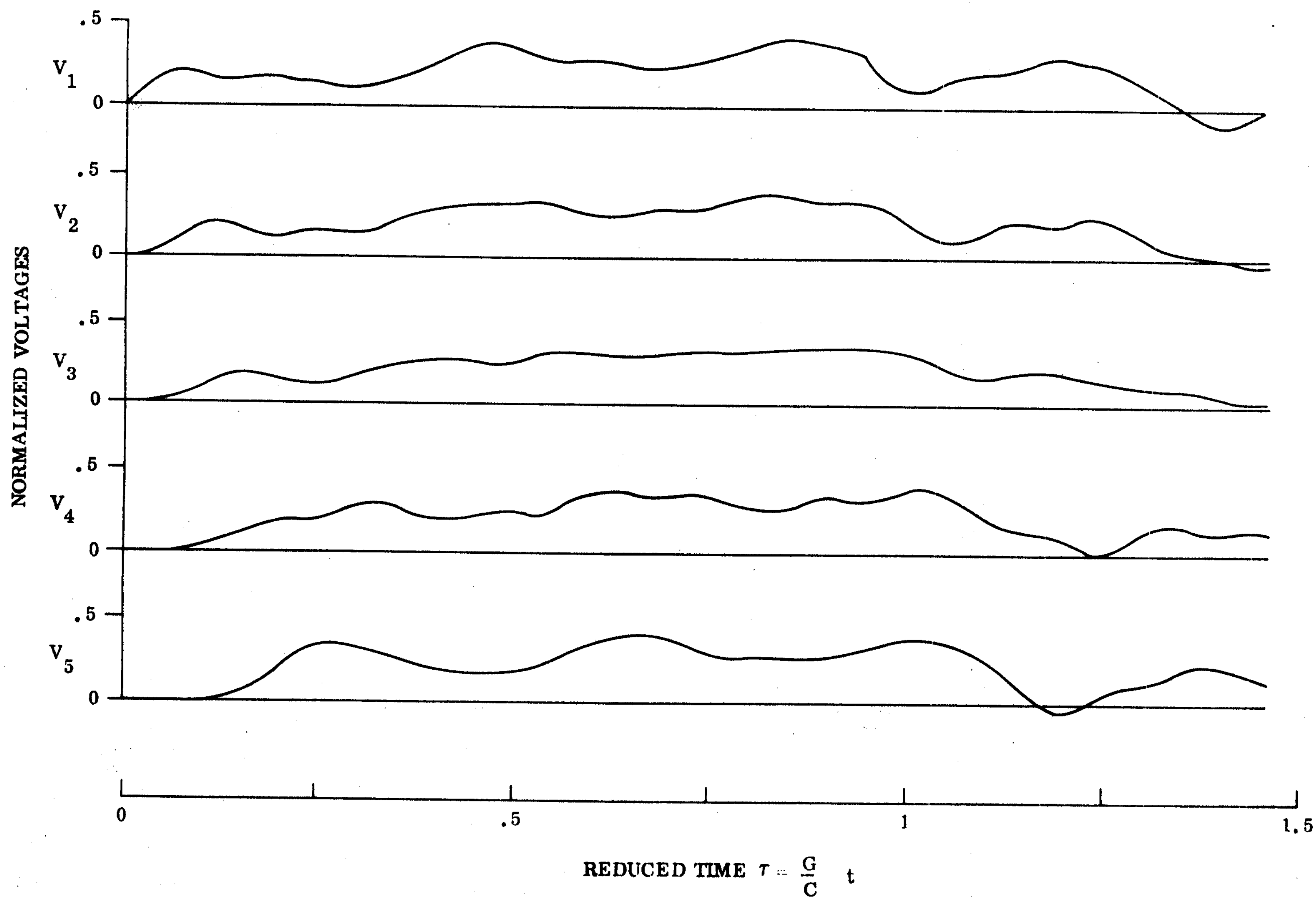

NORMALIZED VOLTAGES
V1
V2
V3
V4
V5
.5
0
.5
1
1.5
REDUCED TIME $\tau = \frac{G}{C}\ t$

FIGURE 8 A

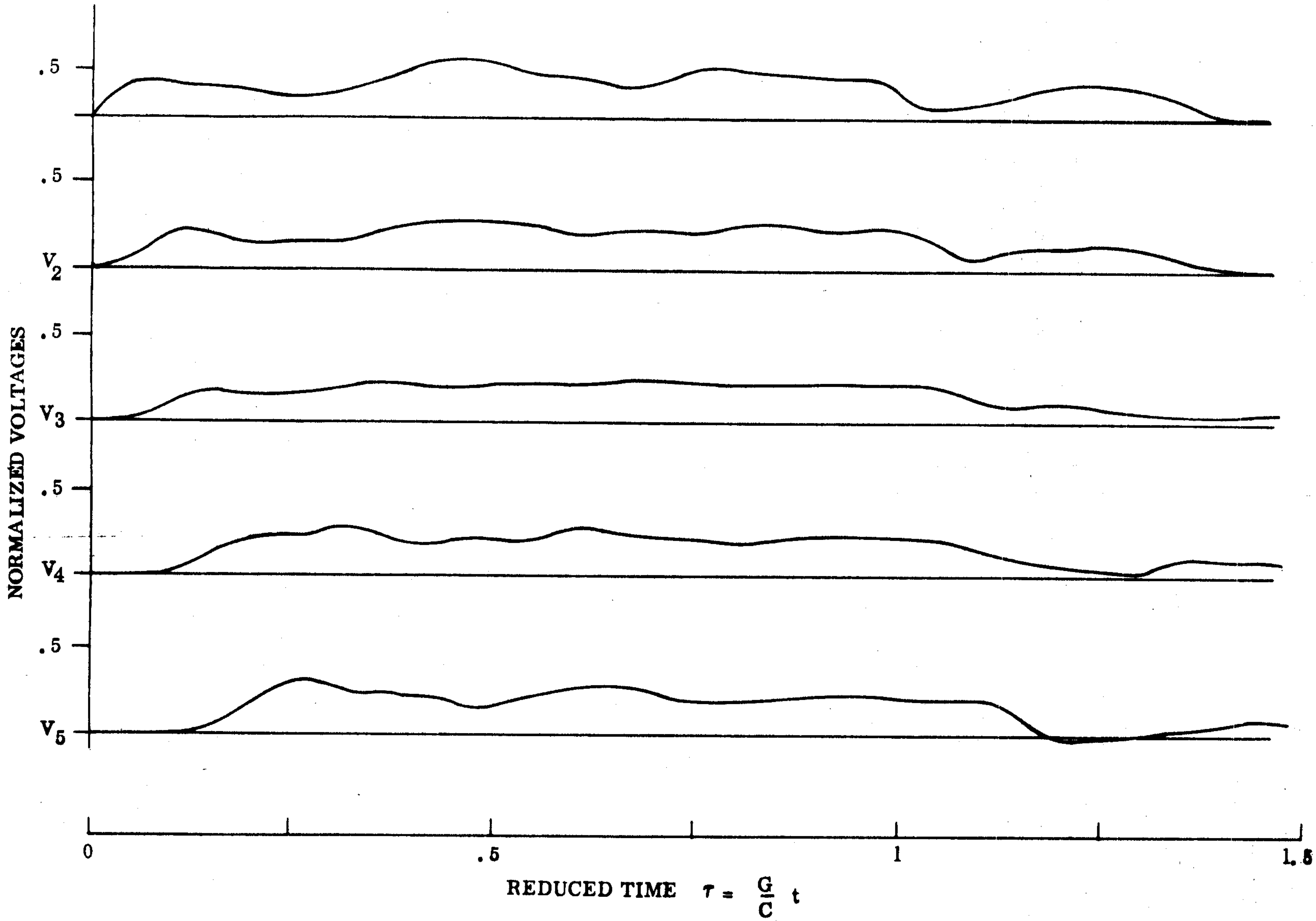
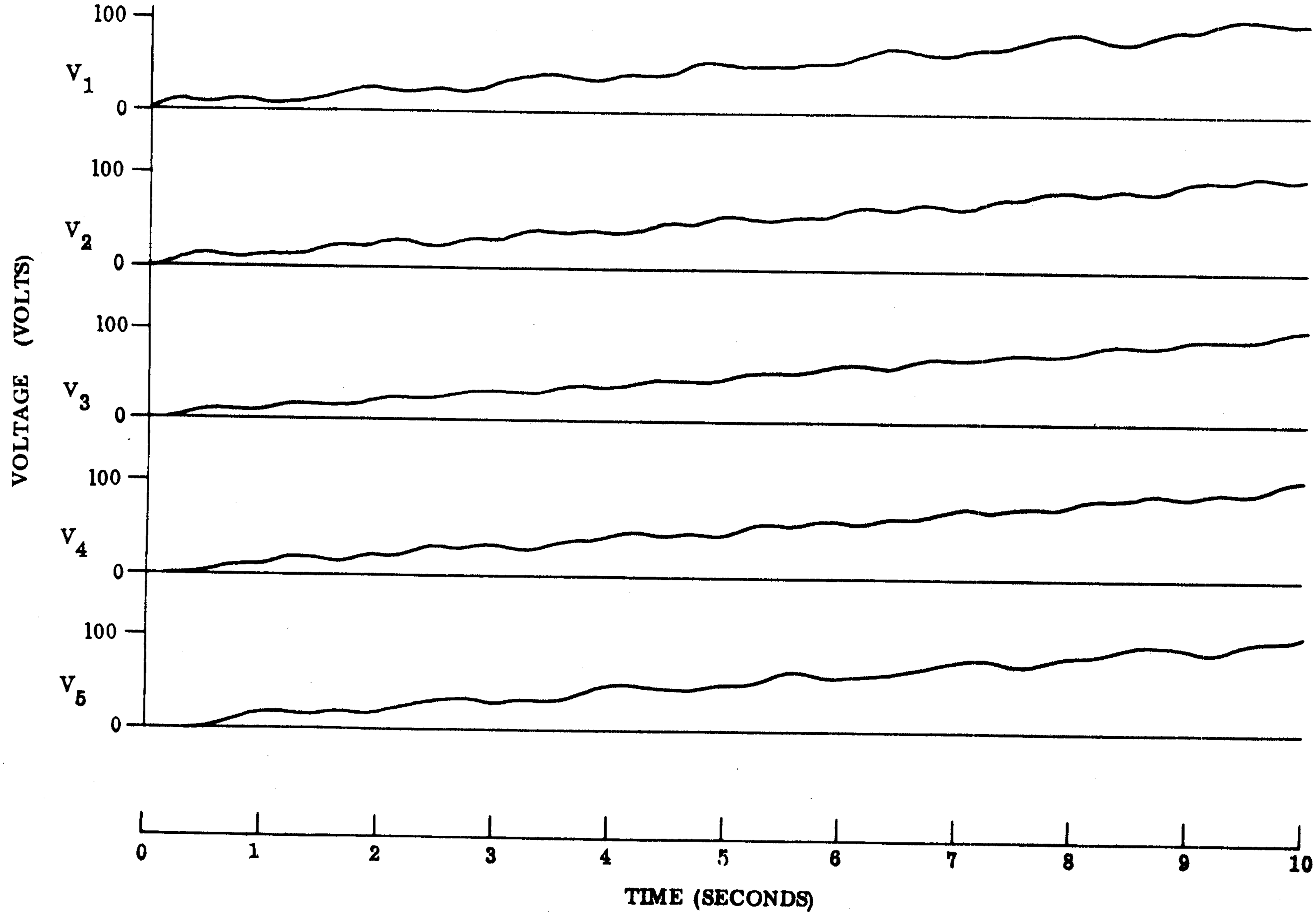
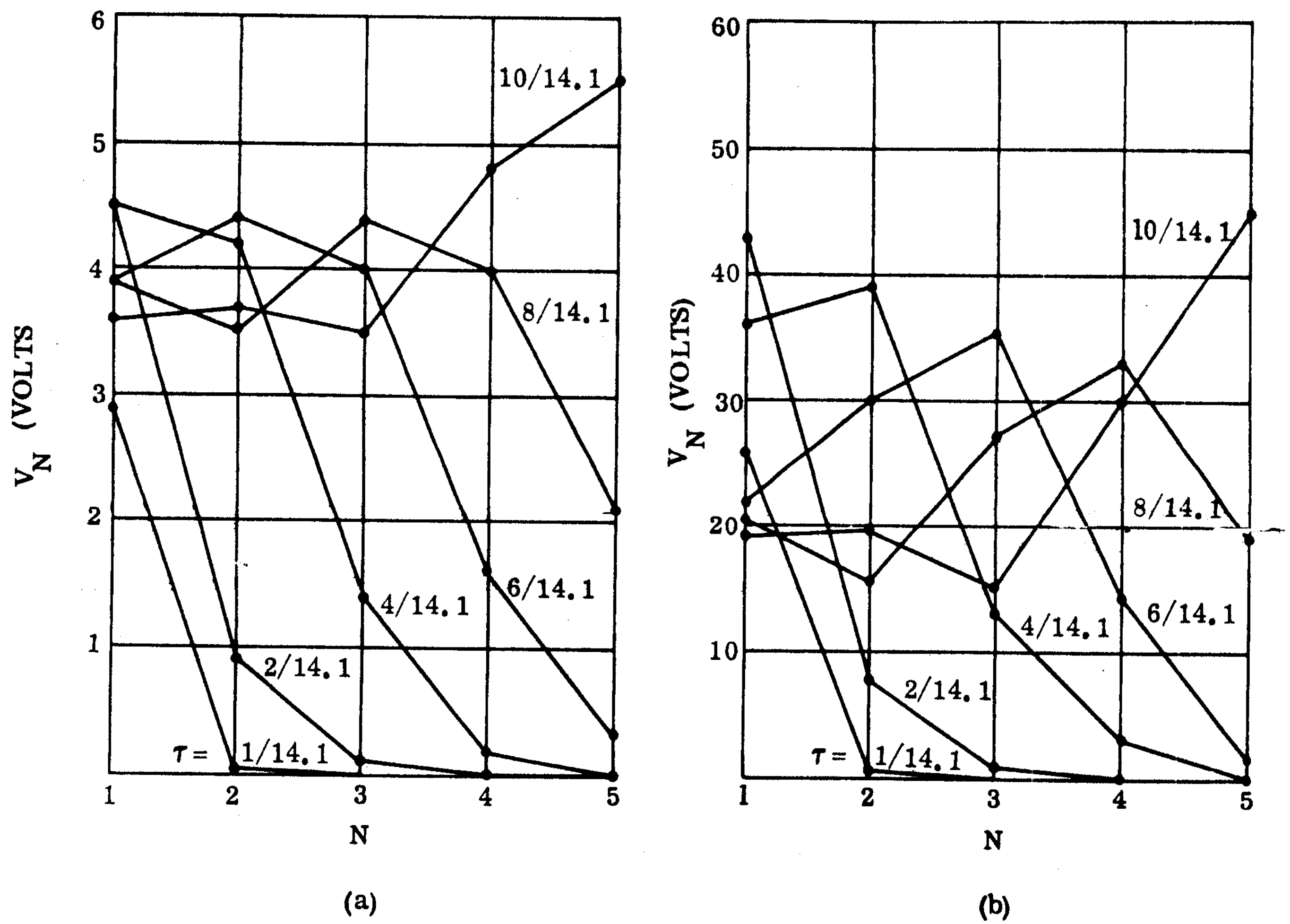

.5
.5
V_2
.5
V_3
.5
V_4
.5
V_5
NORMALIZED VOLTAGES
0
.5
1
1.5
REDUCED TIME $\tau = \frac{G}{C} t$

FIGURE 8 B



VOLTAGE (VOLTS)
100
V1
0
100
V2
0
100
V3
0
100
V4
0
100
V5
0
0
1
2
3
4
5
6
7
8
9
10
TIME (SECONDS)

FIGURE 9



6
5
4
3
2
1
V_N (VOLTS
10/14.1
8/14.1
6/14.1
4/14.1
2/14.1
$\tau =$ 1/14.1
1
2
3
4
5
N

(a)

60
50
40
30
20
10
V_N (VOLTS)
10/14.1
8/14.1
6/14.1
4/14.1
2/14.1
$\tau =$ 1/14.1
1
2
3
4
5
N

(b)

FIGURE 10